Propriétés métriques des ensembles de niveau des applications différentiables sur les groupes de Carnot

Artem Kozhevnikov

Université Paris-Sud

Faculté des sciences d'Orsay

École doctorale de mathématiques de la région Paris-sud (ED 142)

Laboratoire de mathématique d'Orsay (UMR 8628 CNRS)

THÈSE DE DOCTORAT

Discipline : Mathématiques

par

Artem KOZHEVNIKOV

Propriétés métriques des ensembles de niveau des applications différentiables sur les groupes de Carnot

Date de soutenance : le 29 mai 2015

Composition du jury :	Guy DAVID	(Examinateur)
	Bruno FRANCHI	(Rapporteur)
	Hervé PAJOT	(Rapporteur)
	Pierre PANSU	(Directeur de thèse)
	Sergey VODOPYANOV	(Examinateur)

Abstract

We investigate the local metric properties of level sets of mappings defined between Carnot groups that are horizontally differentiable, i. e. with respect to the intrinsic sub-Riemannian structure. We focus on level sets of mapping having a surjective differential, thus, our study can be seen as an extension of implicit function theorem for Carnot groups.

First, we present two notions of tangency in Carnot groups: one based on Reifenberg's flatness condition and another coming from classical convex analysis. We show that for both notions, the tangents to level sets coincide with the kernels of horizontal differentials. Furthermore, we show that this kind of tangency characterizes the level sets called "co-abelian", i.e. for which the target space is abelian and that such a characterization may fail in general.

This tangency result has several remarkable consequences. The most important one is that the Hausdorff dimension of the level sets is the expected one. We also show the local connectivity of level sets and, the fact that level sets of dimension one are topologically simple arcs. Again for dimension one level set, we find an area formula that enables us to compute the Hausdorff measure in terms of generalized Stieltjes integrals.

Next, we study deeply a particular case of level sets in Heisenberg groups. We show that the level sets in this case are topologically equivalent to their tangents. It turns out that the Hausdorff measure of high-codimensional level sets behaves wildly, for instance, it may be zero or infinite. We provide a simple sufficient extra regularity condition on mappings that insures Ahlfors regularity of level sets.

Among other results, we obtain a new general characterization of Lipschitz graphs associated with a semi-direct splitting of a Carnot group of arbitrary step. We use this characterization to derive a new characterization of co-ablian level sets that can be represented as graphs.

Keywords. Sub-Riemannian geometry, Carnot groups, Heisenberg groups, implicit function theorem, tangent cones, Reifenberg flatness condition, Whitney extension theorem, Lipschitz graphs, intrinsic regular surfaces, Hausdorff dimension, Hausdorff measure, Ahlfors regularity, Area formula, Stieltjes integral, lacunary Fourier series, Lévy area, Hölder continuous curves, rough path theory.

Résumé

Nous étudions les propriétés métriques locales des ensembles de niveau des applications horizontalement différentiables entre des groupes de Carnot, c'est-à-dire différentiable par rapport à la structure sous-riemannienne intrinsèque. Nous considérons des applications dont la différentielle horizontale est surjective, et notre étude peut être vue comme une généralisation du théorème des fonctions implicites pour les groupes de Carnot.

Tout d'abord, nous présentons deux notions de tangence dans les groupes de Carnot : la première basée sur la condition de platitude au sens de Reifenberg et la deuxième issue de l'analyse convexe classique. Nous montrons que dans les deux cas, l'espace tangent à un ensemble de niveau coïncide avec le noyau de la différentielle horizontale. Nous montrons que cette condition de tangence caractérise en fait les ensembles de niveau dits "co-abéliens", c'est-à-dire ceux pour lesquels l'espace d'arrivée est abélien, et qu'une telle caractérisation n'est pas vraie en général.

Ce résultat sur les espaces tangents a plusieurs conséquences remarquables. La plus importante est que la dimension de Hausdorff des ensembles de niveau est celle à laquelle l'on s'attend. Nous montrons également la connectivité locale des ensembles de niveau, et le fait que les ensembles de niveau de dimension 1 sont topologiquement des arcs simples. Pour les ensembles de niveau de dimension 1 nous trouvons une formule de l'aire qui permet d'exprimer la mesure de Hausdorff en termes d'intégrales de Stieltjes généralisées.

Ensuite, nous menons une étude approfondie du cas particulier des ensembles de niveau dans les groupes d'Heisenberg. Nous montrons que les ensembles de niveau sont topologiquement équivalents à leurs espaces tangents. Il s'avère que la mesure de Hausdorff des ensembles de niveau de codimension élevée est souvent irrégulière, étant, par exemple, localement nulle ou infinie. Nous présentons une condition simple de régularité supplémentaire pour une application pour assurer la régularité au sens d'Ahlfors des ses ensembles de niveau.

Parmi d'autres résultats, nous obtenons une nouvelle caractérisation générale des graphes Lipschitziens associés à une décomposition en produit semi-direct d'un groupe de Carnot. Nous traitons, en particulier, le cas des groupes de Carnot dont le nombre de strates est plus grand que 2. Cette caractérisation nous permet de déduire une nouvelle caractérisation des ensembles de niveau co-abéliens qui admettent une représentation en tant que graphe.

Mots-clés. Géométrie sous-riemannienne, groupes de Carnot, groupes d'Heisenberg, théorème des fonctions implicites, cônes tangents, condition de platitude de Reifenberg, théorème d'extension de Whitney, graphes lipschitziens, surfaces régulières intrinsèques, dimension de Hausdorff, mesure de Hausdorff, régularité d'Ahlfors, formule de l'aire, intégral de Stieltjes, séries de Fourier lacunaires, aire de Lévy, courbes Hölder continues, théorie de chemins rugueux.

– Ты меня удивляешь, Лев, – объявляет он. – И все вы меня удивляете. Неужели вам здесь не надоело?
– Мы работаем, – возражаю я лениво.
– Зачем работать без всякого смысла?
– Почему же – без смысла? Ты же видишь, сколько мы узнали всего за один день.
– Вот я и спрашиваю: зачем вам узнавать то, что не имеет смысла? Что вы будете с этим делать? Вы все узнаёте и узнаёте и ничего не делаете с тем, что узнаёте.
– Ну, например? – спрашиваю я. [...]
– Например, яма без дна, которую я нашел. Кому и зачем может понадобиться яма без дна?
– Это не совсем яма, – говорю я. – Это скорее дверь в другой мир.
– Вы можете пройти в эту дверь? – осведомляется Щекн.
– Нет, – признаюсь я. – Не можем.
– Зачем же вам дверь, в которую вы все равно не можете пройти?
– Сегодня не можем, а завтра сможем.
– Завтра?
– В широком смысле. Послезавтра. Через год...
– Другой мир, другой мир... – ворчит Щекн. – Разве вам тесно в этом?
– Как тебе сказать... Тесно, должно быть, нашему воображению.
– Еще бы! – ядовито произносит Щекн. – Ведь стоит вам попасть в другой мир, как вы сейчас же начинаете переделывать его наподобие вашего собственного. И конечно же, вашему воображению снова становится тесно, и тогда вы ищете еще какой-нибудь мир и опять принимаетесь переделывать его...

Аркадий и Борис Стругацкие, *Жук в муравейнике*

Table des matières

Remerciements

Je tiens tout d'abord à remercier Sergey Vodopyanov qui fut mon premier directeur de recherche à l'Université de Novossibirsk, et qui m'a introduit à la problématique de la théorie géométrique de la mesure sur les espaces de Carnot.

Je ne pourrais jamais exprimer ce grand remerciement que je dois à Pierre Pansu, mon directeur de recherche qui m'a accompagné dans la Recherche depuis mon entrée au Master 1 de Polytechnique. Tout en me soutenant dans les moments où les problèmes posés devenaient vraiment durs, il a su partager sa richesse de connaissance mathématiques qui allait bien au delà de la géométrie sous-riemannienne. Et en même temps, c'est lui qui au bon moment savait dire qu'il faut arrêter les réflexions et passer à la rédaction, grâce à quoi ce manuscrit a pris sa forme actuelle.

Je remercie Bruno Franchi et Hervé Pajot qui ont accepté d'être rapporteurs pour la thèse. Guy David a encadré mon mémoire de Master 2 et m'a fait découvrir les aspects importants de la théorie de la platitude au sens de Reifenberg. Je suis heureux qu'il ait accepté d'être membre du jury.

Je suis reconnaissant à Bruno Franchi de m'avoir offert la possibilité d'effectuer un séjour de 3 mois à l'université de Bologna qui a été sans doute la période de travail la plus productive.

Je remercie le DMA de l'ENS pour le soutien apporté à ma participation à de nombreuses conférences, ainsi qu'un superbe cadre du travail et des rencontres au cœur de Paris.

Je remercie également mes collègues avec lesquels j'ai eu plaisir à travailler ou être en contact pendant la thèse : Vincent Lafforgue, Assaf Naor, Bruno Franchi, Enrico Le Donne, Zoltan Balogh, Hervé Pajot, Francesco Bigolin, Valentino Magnani.

J'étais très content de participer au séminaire de géométrie sous-riemannienne qui a été organisé à Paris par Davide Barilari et qui m'a toujours donné beaucoup d'inspiration.

Il faut dire aussi qu'à partir de ma deuxième année de thèse j'ai commencé à travailler à mi-temps chez *tinyclues*, où je me suis orienté vers la science des données et ses applications. Je remercie les cofondateurs de *tinyclues*, David Bessis et Jakob Haesler, pour le financement permettant de continuer de travailler sur ma thèse. Je les remercie de leur confiance. Je remercie également toute l'équipe de *tinyclues* pour leur soutien et encouragement.

Je dois beaucoup aux doctorants d'Orsay, en particulier, à mon ami Vladimir Shchur avec qui j'ai passé quelque temps à la Sous-Préfecture de Palaiseau et nous avons compris alors que les problèmes de maths n'étaient pas les plus ardus, et aussi à Imène Hachicha avec qui j'ai encadré le club MATHs.en.JEANS du lycée Berthelot.

Enfin, merci à toi Caroline, pour ton amour et ton soutien sans limite.

1. Introduction

Au sens large ce papier est consacré à l'étude des propriétés métriques des *sous-variétés* dans la géométrie sous-riemannienne. Le travail présenté dans cette thèse s'inscrit dans le programme général de la théorie métrique de la mesure sur les espaces métriques qui a connu un développement actif ces dernières années. Les variétés sous-riemanniennes font partie des espaces métriques ayant gardé beaucoup de structures importantes des espaces euclidiens, or, leurs propriétés métriques peuvent être très différentes, voire surprenantes, comparées à celles des espaces classiques. Voici quelques références particulièrement appréciables [Mon02 ; Gro96 ; FS82 ; AS04 ; CDPT07].

Une variété sous-riemannienne est une variété riemannienne (connexe) $(\mathbb{M}, d)$ munie de champs de vecteurs tangents $\{X_i \subset T\mathbb{M} \mid i = 1, \ldots, n\}$, appelés *horizontaux*. Étant donnés ces champs de vecteurs la métrique sous-riemannienne d qui porte aussi le nom de métrique de *Carnot-Carathéodory* est définie comme

$$\mathsf{d}(a, b) = \inf_{\gamma}\{\mathrm{Length}_d(\gamma)\},$$

où l'infimum est pris sur l'ensemble des courbes (appelées *admissibles* ou *horizontales*) absolument continues $\gamma\colon [0,1] \to \mathbb{M}$ qui relient les deux points, $a = \gamma(0)$ et $b = \gamma(1)$, et dont le vecteur tangent $\gamma'(t)$ appartient au plan $H\mathbb{M}_{\gamma(t)} = \mathrm{span}\{X_i(\gamma(t))\}_{i=1,\ldots,n}$ pour presque tout $t \in [0, 1]$. Le choix de champs de vecteurs $\{X_i\}$ n'est pas arbitraire en géométrie sous-riemanniennes, on suppose notamment que $\{X_i\}$ forment un système non-holonome. Cela veux dire qu'il existe un entier N telle que $H_N\mathbb{M} = T\mathbb{M}$ où les sous-espaces tangents $H_i\mathbb{M}$ sont définis par récurrence

$$H_1\mathbb{M} = H_1\mathbb{M}, \quad H_{i+1}\mathbb{M} = H_i\mathbb{M} + [H_1\mathbb{M}, H_i\mathbb{M}].$$

Si deux sous-espaces tangents sont engendrés par des champs des vecteurs $T_1 = \mathrm{span}\{Y_1, \ldots, Y_k\}$ et $T_2 = \mathrm{span}\{Z_1, \ldots, Z_l\}$, le sous-espace tangent $[T_1, T_2]$ est défini comme $[T_1, T_2] = \mathrm{span}\{[Y_i, Z_j] \mid i = 1, \ldots, k, \quad i = j, \ldots, k\}$.

Cette propriété est en fait suffisante pour garantir que pour toute paire de points $a, b \in \mathbb{M}$ peut être liée par une courbe horizontale et les topologies induites sur $\mathbb{M}$ par d et d sont équivalentes. Le résultat correspondant a été démontré indépendamment par P. K. Rashevsky [Ras38] et W. L. Chow [Cho40]. Selon le domaine, cette propriété peut apparaître sous des noms différents comme, par exemple, condition de Hörmander, propriété de Rashevsky-Chow, non-holonomie ou non-intégrabilité complète, “bracket generating condition”.

Dans ce travail nous allons considérer seulement un modèle local[1] qui correspond à un espace tangent à une variétés sous-riemanniennes en point régulier. Ce modèle a

[1] Comme cela sera clair plus bas notre étude a toujours un caractère local et nous espérons que nos résultats trouverons leur analogues sur des variétés sous-riemanniennes générales.

une structure en plus, - celle de groupe de Lie nilpotent stratifié $(\mathbb{G}, \cdot)$ et il porte le nom de *groupe de Carnot* (voir Section 2.1). La métrique d s'avère être invariante par rapport à l'action (à gauche) du groupe $\mathbb{G}$. Un groupe de Carnot possède aussi d'un groupe d'automorphismes à un paramètre appelés *dilatations* $\{\delta_t\}_{t>0}$. Sur les vecteurs horizontaux l'action induite par les dilatations est juste une multiplication par t comme celle d'une homothétie classique tandis que $\delta_t(X) = t^i X$ si X appartient à i-ième strate. La métrique d est également homogène par rapport à ces dilatations anisotropes.

La notion qui joue un rôle essentiel dans notre étude est celle de la *différentiabilité horizontale*. Par analogie avec la définition classique, une application $F\colon \mathbb{G}^1 \to \mathbb{G}^2$ entre deux groupes de Carnot d est horizontalement différentiable au point $a \in \mathbb{G}^1$ si au voisinage de a elle peut être approximée à l'ordre un en métrique de Carnot par un morphisme homogène $L\colon \mathbb{G}^1 \to \mathbb{G}^2$ (c'est-à-dire, un homomorphisme qui commute avec δ_t voir Definition 2.3.2). Ce morphisme est alors appelé la différentielle horizontale et désigné par $D_hF(a)$. Si $D_hF(\cdot)$ existe et est continu sur un ouvert Ω, alors on dit que $F \in C_h^1(\Omega, \mathbb{G}^2)$. La classe C_h^1 devient tout simplement C^1 si les deux groupes $\mathbb{G}^1 = \mathbb{R}^n$ et $\mathbb{G}^2 = \mathbb{R}^m$ sont abéliens. De façon remarquable, comme pour un calcul différentiel classique, l'appartenance à la classe C_h^1 peut être caractérisée par l'existence de dérivées partielles continues hormis le fait qu'il s'agit maintenant de dérivées le long des champs de vecteurs horizontaux uniquement et le résultat d'une telle dérivation doit être un vecteur horizontal dans l'espace d'arrivée (voir Theorem 2.3.3).

Maintenant nous pouvons parler de l'objet clé de cette thèse : les *sous-variétés* (intrinsèques) dans un groupe de Carnot. L'ensemble $S \subset \mathbb{G}^1$ est appelé une sous-variété intrinsèque si dans un voisinage de tout point $p \in S$ il coïncide avec un ensemble de niveau $F^{-1}(e)$ d'une application horizontalement différentiable $F \in C_h^1(\mathbb{G}^1, \mathbb{G}^2)$ telle que la différentielle horizontale $D_hF(p)$ est surjective[2]. Cette définition est tout à fait semblable à celle d'une sous-variété en géométrie différentielle.

Nous allons étudier les propriétés locales de S ce qui veut dire que nous pouvons considérer S comme étant un ensemble de niveau de F comme ci-dessus dans une petite boule. Nous sommes donc dans une situation similaire à celle du théorème classique des fonctions implicites. Pour les groupes $\mathbb{G}^1$ et $\mathbb{G}^2$ fixes, un exemple trivial de S est donné par $S_0 = L^{-1}(e) \cap B(e, r)$, $r > 0$, une équation "linéaire" avec un morphisme homogène surjectif $L = D_hF(p)\colon \mathbb{G}^1 \to \mathbb{G}^2$. Les questions que nous nous sommes posées sur S sont les suivantes :

- Existe-elle une paramétrisation "canonique" $P\colon S_0 \to S$?
- Quelle est la topologie de S ? Existe-il un homéomorphisme $P\colon S_0 \to S$?
- Quelle est la dimension métrique (de Hausdorff) de S ? $\dim S = \dim S_0$?
- Quelle est la régularité de la mesure de Hausdorff $\mathcal{H}^{\dim S}$ sur S ?
- Comment calculer effectivement cette mesure ?
- Existe-il un espace tangent $\operatorname{Tan}(S, p)$ vers S ?

[2]Bien entendu, pour que S soit non-vide il faut que $\dim H\mathbb{G}^1 \geq \dim H\mathbb{G}^2$.

- Comment caractériser les sous-variétés sans parler des équations F qui les décrivent ?

Ces questions sont d'autant plus difficiles que la régularité de l'application F est faible. En effet, si par exemple $\mathbb{G}^2 = \mathbb{R}^m$ alors pour une application générique $F \in C^1_h$ nous avons grosso modo un comportement C^1 dans les directions horizontales et seulement des estimations de type de Hölder pour les directions de strates supérieures, ce qui est clairement plus faible que C^1.

Le théorème des fonctions implicites classique dit que si $\mathbb{G}^1 = \mathbb{R}^n$ et $\mathbb{G}^2 = \mathbb{R}^m$ alors l'ensemble de niveau $S = F^{-1}(0) \cap B(0,r)$ peut être représenté comme un graphe (d'une application $P \in C^1$) associé à une décomposition de l'espace tangent en $\mathbb{R}^n = \operatorname{Ker} dF(0) \oplus \mathbb{R}^{n-m}$, où la différentielle $dF(0)$ restreinte sur le deuxième facteur $\mathbb{R}^m$ est un isomorphisme. Ainsi, l'étude locale de S se trivialise beaucoup grâce à l'existence d'une paramétrisation P ayant de bonnes propriétés. Le point de départ de notre travail était une généralisation de ce théorème classique sur les groupes de Carnot.

Theorem ([Mag13]). *Etant donnés deux groupes de Carnot $(\mathbb{G}^1, \mathsf{d}^1)$ et $(\mathbb{G}^2, \mathsf{d}^2)$, considérons une application $F \in C^1_h(\Omega, \mathbb{G}^2)$ définie sur un ouvert $\Omega \subset \mathbb{G}^1$. Supposons que la différentielle horizontale $D_hF(a)\colon \mathbb{G}^1 \to \mathbb{G}^2$ est surjective en un point $a \in \Omega$. Supposons en outre que le noyau $\mathbb{K} = \operatorname{Ker} D_hF(a)$ admet un sous-groupe homogène complémentaire $\mathbb{H}$ (ce qui veut dire que $\mathbb{G} = \mathbb{K} \ltimes \mathbb{H}$ est un produit semi-direct et nous pouvons parler d'une décomposition associée $a = a_{\mathbb{K}} \cdot a_{\mathbb{H}}$). Alors il existe un voisinage $U_{\mathbb{K}} \subset \mathbb{K}$ de $a_{\mathbb{K}}$ et un voisinage $U_{\mathbb{H}} \subset \mathbb{H}$ de $a_{\mathbb{H}}$ tel que localement l'ensemble de niveau $F^{-1}(F(a))$ est un $\mathbb{K}\mathbb{H}$-graphe donné par une application continue $P\colon U_{\mathbb{K}} \to U_{\mathbb{H}}$, c'est-à-dire*

$$F^{-1}\big(F(a)\big) \cap U_{\mathbb{K}} \cdot U_{\mathbb{H}} = \{\mathbf{y} \cdot P(\mathbf{y}) \mid \mathbf{y} \in U_{\mathbb{K}}\}.$$

De plus, il existe une constante $C > 0$ telle que

$$\mathsf{d}^1\big(P(\mathbf{y}), P(\mathbf{y}')\big) \leq C\mathsf{d}^1\big(\mathbf{y} \cdot P(\mathbf{y}'), \mathbf{y}' \cdot P(\mathbf{y}')\big).$$

En particulier, l'application P est de classe de Hölder $\mathrm{Hol}^{\frac{1}{m}}\big((U_{\mathbb{K}}, \|\cdot\|), (U_{\mathbb{H}}, \mathsf{d}^1)\big)$, où m est la profondeur de $\mathbb{G}^1$.

Notons que l'exemple le plus simple et le plus étudié où les hypothèses de ce théorème sont réunies est le cas d'une application scalaire, $\mathbb{G}^2 = \mathbb{R}$ (S est alors appelé une hypersurface régulière). De nombreux résultats exploitant cette hypothèse d'existence d'un complémentaire peuvent été retrouvés dans d'autres ouvrages. Citons à titre d'exemple [FSS03c] pour les hypersurfaces dans les groupes d'Heisenberg, [FSS05 ; FSS07] pour les graphes de codimension supérieure dans les groupes d'Heisenberg, [CM06a] pour les hypersurfaces des variétés sous-riemanniennes, [Koz10] pour l'analogue d'une décomposition en produit semi-direct sur des variétés sous-riemanniennes.

L'existence d'une telle paramétrisation permet de répondre plus aisément à beaucoup de questions sur S. Tout d'abord, la continuité de P entraîne que topologiquement S et S_0 sont équivalents (localement). Ensuite, bien que l'application P ne soit pas lipschitzienne en général ([Vit08, Th. 4.35]), le volume (la mesure de Haar sur $\mathbb{K}$) de la projection sur $\mathbb{K}$ d'une boule $B(r) \subset \mathbb{G}^1$ est de l'ordre de $r^{\dim \mathbb{K}}$ (à un facteur multiplicatif prés). D'où on peut déduire facilement que la dimension de Hausdorff est celle qu'on attend, $\dim S = \dim \mathbb{K}$, et que la mesure de Hausdorff $\mathcal{H}^{\dim S} \llcorner S$ est Alhfors régulière. De plus, il est possible d'exprimer la densité (qui est en fait continue) de $P_\#(\mathcal{H}^{\dim S} \llcorner S)$ par rapport à $\mathcal{H}^{\dim S} \llcorner K$ en terme des dérivées horizontales de F.

Il est clair que l'hypothèse de l'existence d'une telle décomposition en un produit semi-direct est assez restrictive. En effet, on voit apparaître un nouveau phénomène algébrique propre à la *géométrie non-commutative*. Étant donné un morphisme surjectif $L\colon \mathbb{G}^1 \to \mathbb{G}^2$, il n'est pas toujours possible de trouver le facteur complémentaire $\mathbb{H}$. Autrement dit, dans la suite exacte de groupes

$$\{0\} \longrightarrow \operatorname{Ker} L \longrightarrow \mathbb{G}^1 \xrightarrow{L} \mathbb{G}^2 \to \{0\}$$

le morphisme $L\colon \mathbb{G} \to \mathbb{H}$ n'est pas toujours scindé à la différence du cas des groupes abéliens $\mathbb{R}^n$ et $\mathbb{R}^m$. Notre premier objectif était de voir ce qui peux se produire en cas d'absence du groupe complémentaire.

Dans le Chapter 3 nous présentons des résultats très généraux sur les ensembles de niveau. Grosso modo, nous démontrons qu'un ensemble de niveau S admet en tout point $p \in S$ un espace tangent unique et continu en p, à savoir $\operatorname{Ker} D_h F(p)$, c'est-à-dire celui qu'on attendait. Pour définir l'espace tangent nous avons utilisé deux approches différentes en parallèle.

Dans la première, nous considérons une condition de platitude de Reifenberg (les travaux [Rei60 ; DKT01 ; DT99] nous servent de références). L'ensemble ϵ-Reifenberg plat S peut être approximé en tout $p \in S$ et pour toute échelle $r > 0$ par un ensemble-modèle $W_{p,r}$ appartenant à une certaine famille (comme des sous-groupes distingués dans notre cas) avec une erreur $\leq \epsilon r$ en métrique de Hausdorff, c'est-à-dire

$$\operatorname{dist}_{\mathsf{d}}\left(B(p,r) \cap S, B(p,r) \cap W_{p,r}\right) \leq \epsilon r.$$

Notons que par rapport à cette formulation classique, nous allons considérer une condition de platitude plus forte qui dit que l'ensemble approximatif $W_{p,r}$ ne dépend pas de l'échelle r. Cette condition (pour en un seul point $p \in S$) entraîne l'existence d'un espace tangent unique au sens de Gromov-Hausdorff.

La deuxième approche consiste à adapter les notions de cônes tangents et paratangents (voir Definition 3.2.1) de l'analyse convexe classique pour les groupes de Carnot et à exploiter le fait que certains d'entre eux coïncident. D'une certaine façon ces cônes tangents sont souvent plus pratiques notamment de point de vue calculatoire.

Nous démontrons le résultat suivant.

Theorem 1.0.1. *Soient $(\mathbb{G}^1, \mathsf{d}^1)$, $(\mathbb{G}^2, \mathsf{d}^2)$ deux groupes de Carnot et Ω un ouvert de $\mathbb{G}^1$. Supposons que pour une application $F \in C^1_h(\Omega, \mathbb{G}^2)$ la différentielle $D_h F(p)$ est surjective en $p \in \Omega$. Alors il existe un voisinage U de p tel que pour l'ensemble de niveau $S := F^{-1}(F(p))$ les deux propriétés équivalentes sont vérifiées :*

1. *$S \cap U$ est un ensemble ϵ-Reifenberg plat avec $\epsilon \to 0$ lorsque l'échelle se raffine par rapport aux noyaux de la différentielle horizontales de F. Autrement dit, il*

existe une fonction $\epsilon\colon (0,\infty) \to (0,\infty)$, $\epsilon(t) \to 0+$ *avec* $t \to 0+$ *telle que pour tout* $a \in U \cap S$

$$\operatorname{dist}_{\mathsf{d}_1}\big(B(a,r)\cap S, B(a,r)\cap W_a\big) \leq \epsilon(r)r, \quad r>0,$$

avec $W_a = a \cdot \operatorname{Ker} D_h F(a)$.

2. *Les quatre cônes tangents coïncident en tout point* $a \in S \cap U$

$$\mathrm{pTan}^+_{\mathbb{G}^1}(S,a) = \mathrm{Tan}^+_{\mathbb{G}^1}(S,a) = \mathrm{Tan}^-_{\mathbb{G}^1}(S,a) = \mathrm{pTan}^-_{\mathbb{G}^1}(S,a) = \operatorname{Ker} D_h F(a).$$

Nous démontrons d'abord la ϵ-Reifenberg platitude dans le Theorem 3.1.1 et en deuxième temps nous démontrons l'équivalence des deux propriétés dans le Theorem 3.3.2. Pour montrer la platitude au sens de Reifenberg il faut montrer deux choses. La première est que pour tout point de $S \cap B(a,r)$ il existe un point proche sur W_a. C'est en fait une conséquence relativement facile de la différentiabilité horizontale. Tandis que la deuxième, c'est-à-dire que pour tout point de W_a il existe un point proche sur $S \cap B(a,r)$, nécessite un argument topologique en plus.

Nous nous demandons aussi à quel point le fait d'avoir des espaces tangents caractérise une sous-variété intrinsèque. Pour répondre à cette question il est indispensable de faire appel au Théorème d'extension de Whitney, Theorem 2.3.6. Il permet sous une certaine condition d'étendre une fonction définie sur un sous-ensemble ($F \equiv Const$ sur S dans notre cas) en une fonction globale de la classe C^1_h. La condition est satisfaite lorsque S possède des espaces tangents. La seule contrainte fondamentale qu'on rencontre ici est que ce théorème ne s'applique en général que pour les groupes d'arrivé $\mathbb{G}^2 = \mathbb{R}^N$ abéliens. Ainsi, nous obtenons une réciproque partielle du Theorem 1.0.1 (voir les Theorems 3.1.12 and 3.3.5).

Theorem 1.0.2. *Soit* $S \subset \mathbb{G}$ *un fermé connexe. Les conditions suivantes sont équivalentes :*

1. S *est une sous-variété intrinsèque co-abélienne (c'est-à-dire avec* $\mathbb{G}^2 = \mathbb{R}^N$*) de codimension* N *;*

2. *Les quatre cônes tangents coïncident en tout point* $a \in S$ *:*

$$\mathrm{pTan}^+_{\mathbb{G}}(S,a) = \mathrm{pTan}^-_{\mathbb{G}}(S,a)$$

   et il existe un point $p \in S$ *tel que* $\mathrm{pTan}^+_{\mathbb{G}}(S,p)$ *est un sous-groupe vertical de codimension* N.

3. *Il existe une famille* $\{W_a \mid a \in S\}$ *d'ensembles fermés homogènes telle que* W_p *est un sous-groupe vertical de codimension* N *pour un certain point* $p \in S$ *et pour tout* $S' \Subset S$ *il existe une fonction* $\epsilon\colon (0,\infty) \to (0,\infty)$, $\epsilon(t) \to 0+$ *avec* $t \to 0+$, *telle que*

$$\operatorname{dist}_{\mathsf{d}}\big(B(a,r)\cap S, B(a,r)\cap (a\cdot W_a)\big) \leq \epsilon(r)r, \quad r>0, \quad \forall\, a \in S'.$$

Il est important de remarquer que le théorème précédent ne peut pas être généralisé au cas où $\mathbb{G}^2$ est non-abélien. Dans le Lemma A.2.1 nous montrons pour un exemple (emprunté à [DOW11]) de groupe de Carnot ultrarigide $\mathbb{G}$ que toute application de classe C_h^1 à valeurs dans dans un groupe $\mathbb{G}/\mathbb{K}(Z)$ est en fait affine pourvu que sa différentielle horizontale soit surjective. Ici, $\mathbb{K}(Z)$ est le sous-groupe engendré par un vecteur de degré maximal Z. Ce qui veut dire que la classe des sous-variétés pour cette paire de groupes $\mathbb{G}$ et $\mathbb{G}/\mathbb{K}(Z)$ est réduite à des translations de $\mathbb{K}(Z)$. Or, il y a beaucoup d'ensembles qui satisfont la condition de tangence comme nous le montrons dans la Section 3.1.4.

Dans la continuation du Chapter 3, nous déduisons plusieurs conséquences de l'existence d'espaces tangents pour les sous-variétés intrinsèques. Dans le Lemma 3.4.4 nous démontrons que S est connexe par arc. A ce stade-là nous ne sommes pas capables de dire plus sur la topologie de S sauf pour le cas de codimension topologique maximale (c'est-à-dire $\operatorname{Ker} D_hF(p) \approx \mathbb{R}$ topologiquement) où nous démontrons que S est un arc (voir Theorem 3.4.5).

L'autre conséquence non-triviale est l'estimation de la dimension de Hausdorff, que nous obtenons comme conséquence d'un énoncé plus général (Theorem 3.5.6).

Theorem 1.0.3. *Soit $\Omega \subset \mathbb{G}^1$ un ouvert, $F \in C_h^1(\Omega; \mathbb{G}^2)$ et $a \in \Omega$. Supposons que $D_hF(a)$ est surjective. Alors il existe un voisinage U de a dans lequel la dimension de Hausdorff de l'ensemble de niveau $\dim F^{-1}(F(a)) \cap U$ est égale à $\dim \mathbb{G}^1 - \dim \mathbb{G}^2$.*

Nous obtenons ce résultat à comme conséquence d'un énoncé plus général (Theorem 3.5.6) qui est en fait se situe dans un cadre très similaire de celui du travail récent de [DR13]. Notons également que ce résultat ne dit rien sur la régularité de la mesure de Hausdorff $\mathcal{H}^{\dim S} \llcorner S$.

Dans le Chapter 4 nous nous plaçons dans un groupe de Carnot $\mathbb{G}$ qui se décompose comme un produit semi-direct des groupes homogènes $\mathbb{K} \ltimes \mathbb{H}$ où $\mathbb{K}$ est distingué. L'objet clé associé à une telle décomposition est un $\mathbb{K}\mathbb{H}$-graphe S qui se décrit comme

$$S = \{\Phi(\mathbf{y}) = \mathbf{y} \cdot \phi(y) \mid \mathbf{y} \in \Omega\}$$

où ϕ est une application de $\Omega \subset \mathbb{K}$ vers $\mathbb{H}$. L'étude de ces objets est bien sûr motivée par le théorème des fonctions implicites, [FSS05 ; FSS07 ; Mag13], mais aussi par la théorie de la réctifiabilité, [FSC06 ; MSC09 ; FSS10 ; CM09 ; FMS13 ; BCC14 ; CM14].

Dans le Theorem 4.3.1 nous présentons une nouvelle caractérisation des sous-variétés co-abéliennes. Etant donné une application ϕ continue, nous pouvons traduire de façon canonique la condition de la tangence de Theorem 1.0.2 en une séries des conditions (non-linéaires) sur ϕ. Pour écrire ces conditions il faut procéder en deux étapes. Premièrement, soit $\pi_{\mathbb{K}} : \mathbb{G} \to \mathbb{K}$ la projection associée à la décomposition $\mathbb{G} = \mathbb{K} \ltimes \mathbb{H}$. à chaque champ de vecteurs Y invariant à gauche sur $\mathbb{G}$ et tangent à $\mathbb{K}$ à l'origine, nous faisons correspondre le champ de vecteurs projeté $\hat{Y}$ sur $\mathbb{K}$ défini par $\hat{Y}_{\mathbf{y}} = d\pi_{\mathbb{K}}(\Phi(\mathbf{y}))\langle Y \rangle$. Le champs de vecteurs $\hat{Y}_{\mathbf{y}}$ est continu et s'écrit de façon polynomiale en termes de $\mathbf{y}$ et $\phi(\mathbf{y})$ (voir Section 4.4). Pour pouvons alors prendre une ligne intégrale de ce champ, qu'on note $\gamma(t) := \operatorname{Exp}_\phi(tY)(\mathbf{y})$, $\gamma(0) = \mathbf{y}$. La trajectoire γ n'est pas forcement unique (malgré cette notation), mais des choix différents mènerons vers la même conclusion. La seconde étape consiste à regarder le comportement métrique de ϕ le long de γ. Ci-dessous nous donnons l'énoncé précis.

Theorem 1.0.4. *Supposons que $\mathbb{G} = \mathbb{K} \ltimes \mathbb{H}$ est une décomposition d'un groupe de Carnot $\mathbb{G}$ avec $\mathbb{H}$ un sous-groupe horizontal et $\mathbb{K}$ vertical. Soit $\mathcal{S}$ un $\mathbb{K}\mathbb{H}$-graphe donné par une application continue $\phi\colon \Omega \to \mathbb{H}$ définie sur un ouvert $\Omega \subset \mathbb{K}$. Les conditions suivantes sont équivalentes :*

1. *$\mathcal{S} \subset F^{-1}(e)$ est une sous-variété co-abélienne de codimension $\dim \mathbb{H}$ avec $F \in C_h^1(\mathbb{G}, \mathbb{H})$ telle que*

$$\operatorname{Ker} D_h F(a) \cap \mathbb{H} = \{e\}, \quad a \in \mathcal{S}.$$

2. *Pour tout champ de vecteurs $Y \in \mathfrak{k}$ et tout point $\mathbf{y} \in \Omega$ on définit une courbe intégrale $\gamma(t) := \operatorname{Exp}_\phi(tY)(\mathbf{y})$. Selon le degré $\deg Y$ de Y, une des deux conditions suivante est satsifaite par $\phi \circ \gamma$.*

A. $\deg Y = 1$ *: Il existe une application linéaire*

$$w_{\mathbf{y}}\colon \mathfrak{k} \cap \mathfrak{g}_1 \to \mathfrak{h}, \qquad \mathfrak{k} = \log(\mathbb{K}), \quad \mathfrak{h} = \log(\mathbb{H}),$$

continue en $\mathbf{y} \in \Omega$ telle que

$$\frac{d}{dt}(\phi \circ \gamma)(t) = w_{\gamma(t)}\langle Y \rangle, \quad t \in I(\gamma).$$

B. $\deg Y \geq 2$ *: Alors,*

$$\phi \circ \gamma \in \mathrm{hol}^{\frac{1}{\deg(Y)}}(I, \mathbb{H}), \quad \mathbf{y} \in \Omega, \quad I \Subset I(\gamma),$$

où le petit-o dans la définition de la classe d'Hölder (Definition 2.1.12) est uniforme pourvu que $\|Y\| \lesssim 1$ et $\gamma(I) \subset \Omega'$ pour un ensemble $\Omega' \Subset \Omega$ fixé à l'avance.

Cette caractérisation peut être considérée comme l'extension des résultats de [Vit08 ; ASV06 ; BS10b] des groupes d'Heisenberg à des groupes de Carnot généraux. C'est une extension partielle car il s'avère que pour les groupes d'Heisenberg, les conditions du deuxième degré (c'est-à-dire pour le vecteur vertical Z) ne sont pas indispensables et découlent des conditions horizontales (en tout cas en codimension 1, et l'argument de [ASV06, Th. 1.3] semble pouvoir s'adapter aux codimensions supérieures). Nous montrons dans Section 4.5 que les conditions verticales ne peuvent pas être omises pour les groupes de profondeur strictement plus grande de 2.

Outre le fait d'être assez universelle, notre caractérisation, Theorem 1.0.4, facilite le calcul algébrique des champs de vecteurs projetés $\hat{Y}$. Remarquons également que le Theorem 1.0.4 est le premier résultat qui pour une fonction scalaire ϕ décrivant un X-graphe dans $\mathbb{H}^1$ donne directement la différentiabilité de ϕ le long de toutes les courbes intégrales de $\hat{Y} = \partial_y - 4\phi\partial_z$, sans faire appel à un argument d'approximation (comparer avec [LM10]).

Nous obtenons le Theorem 1.0.4 comme une conséquence d'une caractérisation des $\mathbb{KH}$-graphes lipschitziens (locaux) (voir Definition 4.2.2). Ce résultat (Theorem 4.2.16) est en fait même plus facile à formuler car les champs de degré différents sont traités de la même façon et il suffit de remplacer $\mathrm{hol}^{\frac{1}{\deg(Y)}}$ par $\mathrm{Hol}^{\frac{1}{\deg(Y)}}$ (qui pour $\deg Y = 1$ devient Lip). En guise d'application de ce dernier théorème et des théorèmes d'approximation de graphes lipschitziens en codimension 1 ([CMPS14 ; BCC14]) nous déduisons le résultat suivant pour des graphes lipschitziens de codimension 2 dans $\mathbb{H}^2$.

Theorem. *Soit $S \subset \mathbb{H}^2$ un graphe lipschitzien par rapport à la décomposition $\mathbb{H}^2 = \mathbb{K} \ltimes \mathbb{H}$ où $\mathbb{H}$ est un sous-groupe horizontal de dimension 2, $\dim \mathbb{H} = 2$. Supposons que $\pi_{\mathbb{K}}(S) = \Omega$ est un ouvert de $\mathbb{K}$. Pour $\mathcal{H}^{2\times 2+2-2}$-presque tout point $p \in S$ la propriété suivante est vraie. Si $\delta_{1/r_i}(p^{-1} \cdot S) \to W$ converge localement en métrique de Hausdorff pour $\{r_i\}_{i\geq 0} \subset \mathbb{R}_+$, $r_i \to 0$, alors W est un plan vertical.*

Cet énoncé ne dit pas que des suites différentes de $\{r_i\}$ résulteront en un même plan vertical W.

Le Chapter 5 est probablement le plus élaboré et contient le plus de détails car c'était le point de départ de cette thèse. Il est consacré à l'étude de l'exemple modèle le plus simple d'une sous-variété intrinsèque pour laquelle l'espace tangent n'admet pas de complémentaire algébrique. Cet exemple est donné par une ligne de niveau d'une application $F \in C^1_h$ définie sur le groupe d'Heisenberg $\mathbb{H}^n$ et à valeurs dans $\mathbb{R}^{2n}$. Notons en plus que $\operatorname{Ker} D_h F(a)$ ne dépend pas de a et est égal en tout point au sous-groupe vertical $\mathbb{Z} = \{\exp(tZ) \mid t \in \mathbb{R}\}$. Nous résumons une partie des résultats obtenus ci-dessous.

Theorem 1.0.5. *Soit $F \in C^1_h(\mathbb{H}^n, \mathbb{R}^{2n})$ telle que $F(0) = 0$ et la différentielle horizontale $D_h F(0)$ est surjective. Alors il existe un voisinage compact U de l'élément neutre $0 \in \mathbb{H}^n$ tel que les propriétés suivantes sont satisfaites par $\Gamma := F^{-1}(0) \cap U$:*

1. *L'ensemble Γ est un arc simple (topologiquement) qui est ε-Reifenberg plat par rapport à $\mathbb{Z}$ avec $\varepsilon \to 0$ uniformément lorsque l'échelle se raffine. (Lemma 5.2.14 and Theorems 5.3.5 and 5.3.7).*

2. *La dimension euclidienne de Γ peut prendre toute valeur dans l'intervalle $[1, 2]$ (Lemma 5.5.13).*

3. *La dimension de Hausdorff de Γ vaut 2 et sa mesure de Hausdorff peut être calculée par la formule suivante (Corollary 5.4.16) :*
$$\mathcal{H}^2(\Gamma) = \liminf_{\|\sigma\|\to 0} \sum_{\sigma} \mathsf{d}(a_i, a_{i+1})^2,$$
où $\sigma = \{a_0 < a_1 < \ldots < a_n\}$ désigne une subdivision ordonnée de Γ (a_0 et a_n sont les extrémités de Γ), et $\|\sigma\| = \max_i \mathsf{d}(a_i, a_{i+1})$.

4. *Si, en outre, $F \in C^{1,\alpha}_h(\mathbb{H}^n, \mathbb{R}^{2n})$, $\alpha > 0$, alors Γ est fortement régulière au sens d'Ahlfors, et la formule de l'aire se réécrit (quitte à choisir la bonne orientation)*
$$\mathcal{H}^2(\Gamma) = \int_\Gamma dz + 2\int_\Gamma x\,dy - 2\int_\Gamma y\,dx,$$

où il s'agit d'intégrales de Stieltjes (Lemma 5.5.5).

5. *Il existe, néanmoins, des exemples de lignes de niveau Γ "rugueuses" telles que*
   - $\mathcal{H}^2(\Gamma) = \infty$ *(Example 5.6.16) ;*
   - $\mathcal{H}^2(\Gamma) = 0$ *(Example 5.6.17) ;*
   - Γ *est* 2*-Ahlfors régulière n'admettant pas de densité volumique en tout point (Example 5.6.19).*

Nous appelons cette ligne de niveau Γ une *courbe verticale* par analogie avec son cône tangent. Le point conceptuel le plus dur ici est de comprendre que Γ n'admet pas de paramétrisation canonique. Le fait que Γ soit une courbe injective est purement topologique. Nous devons mentionner que cette propriété topologique a été indépendamment obtenue dans [LM10] pour le cas $n = 1$ par une technique différente (qui consiste à regarder Γ comme l'intersection de deux hypersurfaces) qui ne semble pas pouvoir se généraliser au cas de $n > 1$. Notre démonstration passe par une paramétrisation à la Reifenberg et elle est légèrement différente de l'argument donné dans le Theorem 3.4.5.

Nous étudions très en détail les propriétés métriques de Γ. Pour en donner un cadre un peu plus général, nous avons introduit dans la Section 5.4 la notion d'une courbe plate $\lambda = (I, \kappa)$ que nous voyons comme une quasi-métrique sur un intervalle $I \subset \mathbb{R}$. Par définition d'une courbe plate, pour tout triplet ordonné de points $a \geq b \geq c$,

$$|\kappa(a, b) + \kappa(b, c) - \kappa(a, c)| \leq m(\kappa(a, c))\kappa(a, c),$$

où $m(t) \searrow 0$ avec $t \to 0$ est un module de platitude. L'exemple auquel on pense ici est celui de $\kappa = \mathsf{d}^2 \llcorner \Gamma$. Nous établissons que λ possède une mesure de probabilité doublante asymptotiquement optimale (voir Lemma 5.4.3 and Corollary 5.4.5). D'où on peut déduire que la dimension de Hausdorff de λ égale 1 (Corollary 5.4.8) ainsi que la formule de l'aire pour la mesure de Hausdorff (Lemma 5.4.9). Cette formule se transforme en une intégrale abstraite de Stieltjes à condition de supposer une régularité supplémentaire du module m, notamment que m soit un module de Dini (Lemma 5.4.12). Si c'est le cas, λ est 1-Ahlfors régulière au sens fort. Pour une courbe verticale Γ nous pouvons contrôler m par le biais des modules de continuité des dérivées horizontales de F. Ainsi, une dérivée de classe Hölder, $F \in C_h^{1,\alpha}$, entraîne la régularité de $\mathcal{H}^2 \llcorner \Gamma$. Notons que nous obtenons une généralisation de la formule de l'aire pour Γ qui était déjà connue ([Mag04 ; Jea06])

$$\mathcal{H}^2(\Gamma) = \int_\Gamma \theta,$$

où θ est une forme de contact sur $\mathbb{H}^n$.

L'existence des courbes verticales irrégulières est un phénomène assez surprenant qui n'était pas observé auparavant ni dans $\mathbb{R}^n$ ni dans les groupes de Carnot. Dans ce contexte il serait intéressant de savoir s'il existe toujours des lignes de niveau co-abéliennes irrégulières, sous des hypothèses algébriques sur les cônes tangents. Par exemple, si on prend une application $F \in C_h^1(E^4, \mathbb{R}^2)$ où E^4 est un groupe d'Engel de dimension 4 (Section 2.2.2), possède-t-elle des lignes de niveau irrégulières ?

Pour construire des exemples de courbes verticales ayant des propriétés prescrites, nous utilisons, en sus du théorème de Whitney, deux autres ingrédients. Le premier ingrédient est une caractérisation exacte des projections des courbes verticales sur le plan horizontal (Eq. (5.38) et Lemma 5.6.12). L'idée d'utiliser une variation généralisée dans cette caractérisation s'inspire notamment de la théorie de chemins rugueux qui a connu un développement majeur ces dernières années (on renvoie le lecteur à [LQ02 ; LCL07] pour ses fondements). Pour les applications $F \in C_h^{1,\alpha}$, cela prend une forme plus compacte. Ainsi, une courbe verticale Γ est une ligne de niveau d'une application de classe $C_h^{1,\alpha}(\mathbb{H}^n, \mathbb{R}^{2n})$, $\alpha > 0$, si et seulement si sa projection $\pi(\Gamma)$ sur le plan horizontal est $\frac{1+\alpha}{2}$-hölderienne en distance euclidienne (Lemma 5.5.11).

Le deuxième ingrédient est le calcul des intégrales de Stieltjes associées à l'aire de Lévy $\int_\gamma x\,dy - y\,dx$ dans le plan. Pour construire des exemples irréguliers nous devons produire des estimations très précises pour les sommes des Stieltjes dans le cas où la régularité de la courbe plane γ est assez faible (typiquement $\gamma \notin \bigcup_{\alpha>1/2} \mathrm{Hol}^\alpha$), ce qui sort du cadre du théorème d'existence classique de Young-Kondurar (Theorem A.1.3). Ces estimations ont été obtenues pour des courbes planes qui s'écrivent comme des séries de Fourier lacunaires (voir Propositions 5.6.5 to 5.6.7). Les séries lacunaires sont en effet très commodes pour ce type d'estimations en raison de leur nature autosimilaire et des interactions faibles entre les différentes fréquences. Ainsi, en jouant de façon délicate sur la vitesse de divergence (ou convergence) des aires de Lévy pour ces séries lacunaires nous sommes capables de construire une courbe verticale avec des irrégularités souhaitées.

Dans le Chapter 6 nous considérons l'ensemble des lignes de niveau $\{\Gamma_p = F^{-1}(p)\}_{p\in\mathbb{R}^{2n}}$ d'une application $F \in C_h^1(\mathbb{H}^n, \mathbb{R}^{2n})$ dont la différentielle est surjective. Nous utilisons les propriétés de courbes verticales, la continuité des Γ_p en métrique de Hausdorff (voir Propositions 3.1.8 and 6.1.3 and Corollary 3.1.9) et un théorème classique de sélection continue pour montrer que les fibres de F forment un feuilletage continu de $\mathbb{H}^n$. Plus précisément nous obtenons le résultat topologique suivant (Theorem 6.1.1).

Theorem 1.0.6. *Soit $F \in C_h^1(\mathbb{H}^n, \mathbb{R}^{2n})$ telle que $F(0) = 0$ et la différentielle horizontale $D_hF(0)$ est surjective. Alors il existe un homéomorphisme*

$$[0,1] \times [-\delta, \delta]^{2n} \ni (t,p) \longrightarrow \Gamma_p(t) \in U \subset \mathbb{H}^n, \quad \delta > 0,$$

sur un voisinage U de $0 \in \mathbb{H}^n$ tel que for tout $p \in [-\delta, \delta]^{2n}$

1. *$\Gamma_p([0,1]) = U \cap F^{-1}(p)$;*
2. *la paramétrisation $\{[0,1] \ni t \to \Gamma_p(t)\}$ induit sur la courbe verticale $U \cap F^{-1}(p)$ une mesure doublante asymptotiquement optimale (voir Lemma 5.4.3).*

Encore une fois nous soulignons le fait qu'une telle paramétrisation n'est pas unique ni canonique.

Une conséquence immédiate de ce résultat est que la topologie locale des sous-variétés de codimension supérieure est la même que celle de leurs cônes tangents (Corollary 6.2.1).

Theorem 1.0.7. *Soit $F \in C_h^1(\mathbb{H}^n, \mathbb{R}^k)$, $1 \leq k \leq 2n$ telle que $F(0) = 0$ et la différentielle horizontale $D_h F(0)$ est surjective. Alors il existe un voisinage U de $0 \in \mathbb{H}^n$ telle que $F^{-1}(0) \cap U$ est homéomorphe à $\operatorname{Ker} D_h F(0)$.*

Nous montrons que ces sous-variétés peuvent être également irrégulières dès que la codimension est dans l'intervalle $[n+1, 2n]$ (Corollary 6.2.4). Il est en effet suffisant de prendre une sous-variété feuilletée par des courbes verticales irrégulières.

A la fin du chapitre, nous montrons la formule de la co-aire (voir [Fed69] pour le cas euclidien) pour les applications $F \in C_h^{1,\alpha}(\mathbb{H}^n, \mathbb{R}^{2n})$, donc sous une hypothèse de régularité supplémentaire. La formule de la co-aire est connue pour les applications lisses. De façon générale, si nous prenons une approximation de l'application F par des applications lisses F_n, alors les lignes de niveau $F_n^{-1}(p)$ vont converger localement vers $F^{-1}(p)$ en métrique de Hausdorff. La (equi-)régularité forte d'Ahlfors de la mesure $\mathcal{H}^2$ sur $F_n^{-1}(p)$ permet d'obtenir la convergence en terme de mesure également.

Pour autant que nous le sachions, la formule de la co-aire n'a pas été démontrée pour $F \in C_h^1(\mathbb{H}^n, \mathbb{R}^{2n})$ (ou pour $F \in \operatorname{Lip}(\mathbb{H}^n, \mathbb{R}^{2n})$). Sa validité reste donc une question très intrigante. Si la formule de la co-aire était vérifiée, cela voudrait dire, par exemple, que les fibres irrégulières $F^{-1}(p)$ de mesure nulle n'arrivent que pour un ensemble négligeable de $p \in \mathbb{R}^{2n}$. Notons que par l'inégalité générale de la co-aire, l'ensemble des fibres de mesure infinie est négligeable dans $\mathbb{R}^{2n}$.

2. Basic notions and notations

2.1. Carnot Group

A *Carnot* group [FS82] is a nilpotent connected and simply connected Lie group $(\mathbb{G}, \cdot)$ whose Lie algebra $\mathfrak{g}$ admits a *stratification*, i. e. a direct sum decomposition

$$\begin{aligned} &\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m, \ \dim \mathfrak{g}_1 \geq 2; \\ &[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, \quad 1 \leq i \leq m-1, \quad [\mathfrak{g}_1, \mathfrak{g}_m] = \{0\}. \end{aligned}$$

The brackets $[\cdot, \cdot]$ here stand for commutators on $\mathfrak{g}$. The integer m is called *depth* (*steps* number) of the Carnot group. The left and right translations will be denoted by L_a and R_a respectively: $L_a(b) = a \cdot b$ and $R_a(b) = b \cdot a$. The symbol d represents the differential in the usual sense. For instance, $dL_a(b)\langle V\rangle$ means the differential of L_a in point $b \in \mathbb{G}$ acting on the vector $V \in T_b\mathbb{G}$. A vector field $X \in T\mathbb{G}$ is said to be left-invariant if $X(a) = dL_a(e)\langle X(e)\rangle$, where e denotes the identity element.

Vectors belonging to $\mathfrak{g}_1$ are called *horizontal*. The horizontal bundle $H\mathbb{G}$ is a subbundle of $T\mathbb{G}$ defined by $H_a\mathbb{G} = dL_a(e)\langle \mathfrak{g}_1\rangle$. A smooth submanifold $S \subset \mathbb{G}$ is called horizontal if $TS \subset H\mathbb{G}$.

Theorem. *Let $\mathbb{G}$ be a connected, simply connected, nilpotent Lie group. Then the exponential map $\exp\colon \mathfrak{g} \to \mathbb{G}$ is a global diffeomorphism.*

Notation. The symbol $\log\colon \mathbb{G} \to \mathfrak{g}$ denotes the map inverse to the exponential. For $a \in \mathbb{G}$ the shifted exponent is defined as $\exp(X)(a) := L_a(\exp(X)) = a \cdot \exp(X)$.

Definition 2.1.1. On the Lie algebra $\mathfrak{g}$ we define the one-parametric multiplicative group of automorphisms $\{\delta_t\}_{t>0}$, called *dilatations*: δ_t acts on $\mathfrak{g}_i$ as multiplication by t^i. Via the exponential map $\exp\colon \mathfrak{g} \to \mathbb{G}$ we transfer the action of $\{\delta_t\}_{t>0}$ on $\mathbb{G}$ keeping the same notation.

Notation. The degree $\deg(X)$ of $X \in \mathfrak{g}$ is the smallest integer k such that $X \in \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$. If now $W \subset \mathfrak{g}$, then we define $\deg W := \max\{\deg X \mid X \in W\}$. If $A \subset \mathbb{G}$, then we put $\deg A := \deg \log(A)$.

A Carnot group $\mathbb{G}$ is a homogeneous space of topological dimension $N := \sum_{i=1}^m \dim \mathfrak{g}_i$ and homogeneous dimension $Q := \sum_{i=1}^m i \dim \mathfrak{g}_i$. If $\mathbb{G}$ is noncommutative, then the homogeneous dimension is strictly greater than the topological one.

2.1.1. Exponential coordinates.

A vector $X \in \mathfrak{g}$ is called homogeneous if it lies in some $\mathfrak{g}_k = \delta_t(\mathfrak{g}_k)$. Let's fix a basis $\{X_1, \dots, X_N\}$ in $\mathfrak{g}$ that respects the stratification, i.e. all X_i are homogeneous. This gives rise to coordinates $(x_1, \dots, x_N) \in \mathbb{R}^N \equiv \mathfrak{g}$. We always assume that the vectors X_i are ordered in a such way that their degrees increase. We also define a scalar product $\langle \cdot, \cdot \rangle$ (as well as the corresponding Euclidean norm $\| \cdot \|$) on $\mathfrak{g}$ that makes the basis $\{X_1, \dots, X_N\}$ orthonormal. (This scalar product is not canonical, at the same time, it will not play an important role.)

Using the exponential map we induce the coordinates on $\mathbb{G}$ called "exponential": if $a = \exp(\sum_{i=1}^N x_i(a)X_i)$ then $a \in \mathbb{G}$ is identified with $(x_1(a), \dots, x_N(a)) \in \mathbb{R}^N$. In particular, $e \equiv 0 \in \mathbb{R}^N$ and $a^{-1} \equiv -(x_1(a), \dots, x_N(a))$. The Lebesgue measure $\mathcal{L}^N$ on $\mathbb{R}^N \equiv \mathbb{G}$ is a biinvariant Haar measure and $d(\delta_t \mathcal{L}^N) = t^Q d\mathcal{L}^N$.

According to the *Baker-Campbell-Hausdorff* formula the group operation on $\mathbb{G}$ turns out to be polynomial when written in exponential coordinates. Here, we give Dynkin's combinatorial representation of this formula (see [Dyn47]):

$$\log(\exp X \cdot \exp Y) = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_i+s_i>0 \\ 1\le i\le n}} \frac{\left(\sum_{i=1}^n (r_i+s_i)\right)^{-1}}{r_1! s_1! \cdots r_n! s_n!} [X^{r_1}Y^{s_1}X^{r_2}Y^{s_2}\dots X^{r_n}Y^{s_n}]$$

where for $X, Y \in \mathfrak{g}$ we set

$$[X^{r_1}Y^{s_1}\dots X^{r_n}Y^{s_n}] = [\underbrace{X,[X,\dots[X}_{r_1},\underbrace{[Y,[Y,\dots[Y}_{s_1},\ \dots\ \underbrace{[X,[X,\dots[X}_{r_n},\underbrace{[Y,[Y,\dots Y}_{s_n}]]\dots]].$$

The first few terms of its expansion are explicitly given by

$$\log(\exp X \cdot \exp Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] - [Y,[X,Y]]) - \frac{1}{24}[Y,[X,[X,Y]]] \\ + \text{high order commutators.}$$

2.1.2. Metric structure

Definition 2.1.2. An *homogeneous norm* ρ is a continuous function on $\mathbb{G}$ that enjoys the following properties:

- positivity, $\rho(a) \ge 0$, $\rho(a) = 0$ if and only if $a = e$;
- symmetry, $\rho(a^{-1}) = \rho(a)$;
- homogeneity, $\rho(\delta_t(a)) = t\rho(a)$, $t > 0$;
- generalized triangle inequality, $\rho(a \cdot b) \le K(\rho(a) + \rho(b))$, $K \ge 1$.

Example 2.1.3. One can put, for instance,

$$\rho(\exp(V_1 + \dots + V_m)) = \max_{i=1,\dots,m} \|V_i\|^{1/i}, \quad V_i \in \mathfrak{g}_i. \tag{2.1}$$

Remark. Two homogeneous norms on $\mathbb{G}$, ρ' and ρ, are equivalent [FS82], $\rho' \approx \rho$.

Notation. We write $A \lesssim B$ for quantities A and B if there is a constant $\infty > K > 0$ independent of A and B such that $A \le KB$. Sometimes when we want to emphasize the dependence of K on some exterior parameter a, we shall write $A \overset{(a)}{\lesssim} B$. The relation $\gtrsim$ is defined analogously. We write $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$.

Definition 2.1.4. Using a homogeneous norm ρ, we construct the *homogeneous distance* d_ρ on $\mathbb{G}$: we put $\mathsf{d}_\rho(a, b) = \rho(a^{-1} \cdot b)$ for all $a, b \in \mathbb{G}$.

Definition 2.1.5. A set E with a function $d \colon E \times E \to [0, \infty)$ is called a QUASI-METRIC SPACE if the following condition hold:

- $d(z, z') \ge 0$ for every $z, z' \in E$ and $d(z, z') = 0$ if and only if $z = z'$;
- $d(z, z') = d(z', z)$ for every $z, z' \in E$;
- $d(z, z'') \le K(d(z, z') + d(z', z''))$ for every $z, z', z'' \in E$ and some fixed $K \ge 1$.

The function d is called a quasi-metric. If $K = 1$, then (E, d) is a metric space.

Remark 2.1.6. If (E, d) is a metric space, then in general d^p is not a metric on E for $p > 1$, but still a quasi-metric.

Remark. By definition, d_ρ is homogeneous and left invariant. In general, $(\mathbb{G}, \mathsf{d}_\rho)$ is only quasi-metric space unless $K = 1$.

Remark. Most of the time we don't care about a concrete choice of ρ. So, we shall write merely d instead of d_ρ. However, for some special Carnot groups (typically, Heisenberg groups) d can be specified explicitly.

There is still one special example of metric d called Carnot-Carathéodory distance d_{cc} that we should mention here because of its importance in the applications.

Remark. Let $X_1, \dots, X_l \in \mathfrak{g}_1$ be an orthonormal basis of horizontal vector fields. We say that an absolutely continuous curve $\gamma \colon [0, T] \to \mathbb{G}$ is a sub-unit horizontal curve if there exist measurable functions $c_1(s), \dots, c_l(s)$, $s \in [0, T]$, such that $\sum_j c_j^2 \le 1$ and

$$\gamma'(s) = \sum\nolimits_j c_j(s) X_j(\gamma(s)), \qquad \text{for almost every } s \in [0, T].$$

For $a, b \in \mathbb{G}$, the metric $\mathsf{d}_{cc}(a, b)$ is given by

$$\mathsf{d}_{cc}(a, b) = \inf\{T > 0 \mid \gamma \colon [0, T] \to \mathbb{G} \ \text{ is sub-unit}, \gamma(0) = a, \ \gamma(T) = b\}.$$

Since $\mathfrak{g}_1$ is bracket-generating the whole tangent space, $\mathsf{d}_{cc}(a, b)$ is finite for any $a, b \in \mathbb{G}$ (this fact is known as Rashevsky-Chow theorem). Moreover, it's easy to check from the definition that d_{cc} is left-invariant and homogeneous.

Notation. $B(a, r) = \{y \in \mathbb{G} \mid \mathsf{d}(a, b) < r\} \Subset \mathbb{G}$ is an open ball in distance d.

Proposition 2.1.7 ([FS82])**.** *Given a Riemannian metric d_{riem} on $\mathbb{G}$, for any compact $K \Subset \mathbb{G}$ there are positive constants C_1 et C_2 such that*

$$C_1 d_{riem}(a, b) \le \mathsf{d}(a, b) \le C_2 d_{riem}(a, b)^{\frac{1}{m}}, \quad a, b \in K.$$

In particular, d and d_{riem} define the same topology.

Hausdorff Measure. Let (X, d) be a quasi-metric space. We denote by $\operatorname{diam} E = \sup\{\mathsf{d}(a,b) \mid a, b \in E\}$ the diameter of set $E \subset X$. Let $0 \le k \in \mathbb{R}$. For $\varepsilon > 0$ and $E \subset X$ we define

$$\mathcal{H}^k_\varepsilon(E) = \inf\left\{ \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^k \mid E \subset \bigcup_{i=1}^{\infty} E_i, \operatorname{diam} E_i \le \varepsilon \right\},$$

$$\mathcal{S}^k_\varepsilon(E) = \inf\left\{ \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^k \mid E \subset \bigcup_{i=1}^{\infty} E_i, \operatorname{diam} E_i \le \varepsilon, E_i \text{ is a ball} \right\}.$$

Definition. We define the *k-dimensional Hausdorff measure* $\mathcal{H}^k$ of E as

$$\mathcal{H}^k(E) = \lim_{\varepsilon \to 0+} \mathcal{H}^k_\varepsilon(E) = \sup_{\varepsilon > 0} \mathcal{H}^k_\varepsilon(E),$$

as well as the *k-dimensional spherical Hausdorff measure* $\mathcal{S}^k$ of E as

$$\mathcal{S}^k(E) = \lim_{\varepsilon \to 0+} \mathcal{S}^k_\varepsilon(E) = \sup_{\varepsilon > 0} \mathcal{S}^k_\varepsilon(E).$$

Those two measures, $\mathcal{H}^k$ and $\mathcal{S}^k$, are exterior Borel regular on $\mathbb{G}$ that are mutually comparable. We recall that the *Hausdorff dimension* of E is the number

$$\dim E = \sup\{k \mid \mathcal{H}^k(E) = \infty\} = \inf\{k \mid \mathcal{H}^k(E) = 0\}.$$

Remark. Observe that the different choices of left invariant homogeneous distance d on $\mathbb{G}$ lead to comparable Hausdorff measures. The Hausdorff dimension of $\dim \mathbb{G} = Q$ and, in fact, a biinvariant Haar (volume) measure is proportional to $\mathcal{H}^Q$ on $\mathbb{G}$ and $\mathcal{H}^{\dim \mathbb{G}}(B(a,r)) = C r^{\dim \mathbb{G}}$ for every $a \in \mathbb{G}$, $r \ge 0$.

Definition 2.1.8. Let $\lambda \colon [t_-, t_+] \subset \mathbb{R} \to (E, d)$ be a curve in a quasi-metric space. Then for $p > 0$ the p-variation of λ is defined as

$$\operatorname{Var}^p(\lambda) = \sup\left\{ \sum_{i=0}^{l-1} d(\lambda(t_i), \lambda(t_{i+1}))^p \;\mid\; t_- \le t_0 \le \ldots \le t_l \le t_+ \right\}^{\frac{1}{p}}.$$

Remark. If d is a metric and λ is continuous and injective then $\operatorname{Var}^1(\lambda) = \mathcal{H}^1(\lambda([t_-, t_+]))$.

Definition. We extend the distance d to a distance between point $a \in \mathbb{G}$ and set $E \subset \mathbb{G}$ by putting

$$\mathsf{d}(a, E) = \inf\{\mathsf{d}(a,b) \mid b \in E\}.$$

Definition 2.1.9. We define *Hausdorff distance* $\operatorname{dist}_{\mathsf{d}}$ between $E_1, E_2 \subset X$ by

$$\operatorname{dist}_{\mathsf{d}}(E_1, E_2) = \max\big\{ \sup_{a \in E_2} \mathsf{d}(a, E_1), \sup_{b \in E_1} \mathsf{d}(b, E_2) \big\}.$$

Restrict on compact subsets, the Hausdorff distance $\operatorname{dist}_{\mathsf{d}}$ has all properties of metric. Moreover, if X is compact, then the family of all compact subsets of X equipped with the Hausdorff distance is a compact metric space.

Definition 2.1.10. A Borel measure μ on X is *doubling* if there is a constant $C \geq 1$ such that for any metric ball $B(a,r) \subset X$

$$\mu(B(a,r)) \leq C\mu(B(a,2r)).$$

Definition 2.1.11. A Borel measure μ on X is *k-Ahlfors regular* (of dimension $k > 0$) if there are constants $r_0, C > 0$ such that for any $a \in \operatorname{supp}(\mu)$ and any $0 < r \leq r_0$

$$C^{-1} r^k \leq \mu(B(a,r)) \leq C r^k.$$

Definition 2.1.12. A map $f\colon (X, d_X) \to (Y, d_Y)$ between two quasi-metric space is Hölder continuous of exponent $\beta > 0$, $f \in \mathrm{Hol}^\beta(X,Y)$, if

$$\|f\|_{\mathrm{Hol}^\beta} := \sup_{d_X(a,b)>0} d_Y(f(a), f(b)) d_X(a,b)^{-\beta}$$

is finite. A map $f \in \mathrm{hol}^\beta(X,Y) \subset \mathrm{Hol}^\beta(X,Y)$ if for any compact set $K \Subset X$

$$d_Y(l(a), l(b)) = o(d_X(a,b)^\beta), \quad a, b \in K, \quad d_X(a,b) \to 0.$$

The space $\mathrm{Hol}^1(X,Y)$ is also called the space of *Lipschitz* functions and denoted by $\mathrm{Lip}(X,Y)$.

The following general result is called “coarea inequality” that is also known as Eilenberg’s inequality.

Theorem 2.1.13 ([Fed69, Th. 2.10.25])**.** *If $f\colon X \to Y$ is a Lipschitz map between metric spaces, $A \subset X$, $0 \leq k, m < \infty$, then*

$$\int \mathcal{H}^k(A \cap f^{-1}(p))\, d\mathcal{H}^m(p) \overset{(k,m)}{\lesssim} \mathrm{Lip}(f)^m\, \mathcal{H}^{k+m}(A),$$

provided either $\{p \mid \mathcal{H}^k(A \cap f^{-1}(p)) > 0\}$ is the union of a countable family of sets with finite $\mathcal{H}^m$ measure, or Y is boundedly compact.

2.2. Some examples of Carnot Groups

2.2.1. Heisenberg groups

The n-th Heisenberg group $\mathbb{H}^n$ is a Carnot group of topological dimension $N = 2n+1$ whose the Lie algebra is of depth $m = 2$: $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Here, $\mathfrak{g}_1$ is of dimension $2n$ and generated by the vectors $X_1, \dots, X_n, Y_1, \dots, Y_n$, whereas $\dim V_2 = 1$ and $V_2 = \operatorname{span}\{Z\}$. Thus, the homogeneous dimension of $\mathbb{H}^n$ equals $Q = 2n+2$. Non-trivial commutation relations are generated by $[X_j, Y_j] = -4Z$. Let us denote $\pi(x,y,z) := (x,y) \in \mathbb{R}^{2n}$ the projection on so-called “horizontal plan”. By Baker-Campbell-Hausdorff formula we get that the group operation on $\mathbb{H}^n = \mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \\ z + z' + 2\mathcal{B}\big((x,y),(x',y')\big) \end{pmatrix},$$

where

$$\mathcal{B}\big((x,y),(x',y')\big) := \langle x', y\rangle_{\mathbb{R}^n} - \langle x, y'\rangle_{\mathbb{R}^n}.$$

The dilatations act by $\delta_t(x,y,z) = (tx, ty, t^2 z)$. The centre of $\mathbb{H}^n$ is denoted by $\mathbb{Z} := \{\exp(tZ) \mid t \in \mathbb{R}\}$. For Heisenberg groups we will use the following homogeneous norm :

$$\rho(x,y,z) = \max\{\sqrt{\|x\|^2 + \|y\|^2}, |z|^{1/2}\}. \tag{2.2}$$

By a straightforward computation one can show that the obtained distance d is, indeed, a metric. Moreover, the diameter of a ball of radius r is equal to $2r$. Remark however, that d is not geodesic metric.

The basis of left invariant vector fields is given by

$$\begin{aligned} &X_i(x,y,z) = \partial_{x_i} + 2y_i\partial_z, \quad Y_i(x,y,z) = \partial_{y_i} - 2x_i\partial_z, \\ &Z(x,y,z) = \partial_z = -\frac{1}{4}[X_i, Y_i], \quad i = 1, \dots, n. \end{aligned}$$

The dual basis of differential forms is

$$w_x = dx, \quad w_y = dy, \quad w_z = dz - 2\sum_{i=1}^{n}(x_i\, dy_{n+i} - y_i\, dx_{n+i}).$$

The form w_z carries the name of *contact form* (as a Heisenberg group carries also a contact structure).

With the coordinates chosen above, the horizontal differential of $f \in C_h^1(\mathbb{H}^n, \mathbb{R})$ can be written as follows

$$D_h f(a)\langle v\rangle = \langle \nabla_{\mathbb{H}^n} f(a), \pi(v)\rangle, \tag{2.3}$$

where $\nabla_{\mathbb{H}^n} f := \big(X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f\big)$ is the *horizontal gradient* of f at a.

2.2.2. Engel group

The Engle group $\mathbb{E}^4$ is a Carnot group of depth $m = 3$, topological dimension $n = 4$ and homogeneous dimension $q = 7$. The decomposition of Lie algebra is given by

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3,$$

where $\mathfrak{g}_1 = \operatorname{span}\{V, X\}$, $\mathfrak{g}_2 = \operatorname{span}\{Y\}$, $\mathfrak{g}_3 = \operatorname{span}\{Z\}$. Non-zero commutation relations are the following

$$[V, X] = Y, \quad [V, Y] = Z.$$

We use the realisation of $\mathbb{E}$ as $\mathbb{R}^4$ via the exponential coordinates (v, x, y, z), so the group operation reads as

$$\begin{pmatrix} v \\ x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} v' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} v + v' \\ x + x' \\ y + y' + \frac{1}{2}(vx' - xv') \\ z + z' + \frac{1}{2}(vy' - yv') + \frac{1}{12}(v - v')(vx' - xv') \end{pmatrix}.$$

The dilatations act by $\delta_t(v,x,y,z) = (tv, tx, t^2y, t^3z)$, $t > 0$. In the chosen coordinates the left invariant vector fields read

$$\begin{aligned}
V(v,x,y,z) &= \partial_v - \tfrac{x}{2}\partial_y - \big(\tfrac{y}{2} + \tfrac{vx}{12}\big)\partial_z,\\
X(v,x,y,z) &= \partial_x + \tfrac{v}{2}\partial_y + \tfrac{v^2}{12}\partial_z,\\
Y(v,x,y,z) &= \partial_y + \tfrac{v}{2}\partial_z,\\
Z(v,x,y,z) &= \partial_z,
\end{aligned}$$

and their dual differential forms are

$$\begin{aligned}
&w_v = dv, \quad w_x = dx, \quad w_y = \tfrac{1}{2}x\,dv - \tfrac{1}{2}v\,dx + dy,\\
&w_z = \big(\tfrac{1}{2}y - \tfrac{1}{6}xv\big)\,dv + \tfrac{1}{6}v^2\,dx - \tfrac{1}{2}v\,dy + dz.
\end{aligned}$$

2.3. Horizontal differentiability

Let $(\mathbb{G}^1, \delta_t^1, \rho_1, \mathsf{d}_1, e_1)$ and $(\mathbb{G}^2, \delta_t^2, \rho_2, \mathsf{d}_2, e_2)$ be two Carnot groups.

Homogeneous homomorphism

Definition. A continuous homomorphism $L\colon \mathbb{G}^1 \to \mathbb{G}^2$ is called *homogeneous* (or *horizontal*), if $L \circ \delta_t^1 = \delta_t^2 \circ L$ for every $t > 0$.

Remark. The space of horizontal homomorphisms $\mathrm{Hom}_h(\mathbb{G}^1, \mathbb{G}^2)$ between $\mathbb{G}^1$ and $\mathbb{G}^2$ is naturally endowed with a group structure and a norm

$$\|L\| = \sup_{a \in \mathbb{G}^1 \setminus e_1} \frac{\rho_2(L(a))}{\rho_1(a)} < \infty.$$

To continuous morphism of Carnot group $L\colon \mathbb{G}^1 \to \mathbb{G}^2$ corresponds the morphism of Lie algebras $\mathcal{L} = \exp_2^{-1} \circ L \circ \exp_1\colon \mathfrak{g}^1 \to \mathfrak{g}^2$. Homogeneity condition reads then $\mathcal{L}(\mathfrak{g}_1^1) \subset \mathfrak{g}_1^2$.

The kernel of homogeneous homomorphism $\operatorname{Ker} L = \mathbb{K}$ is a normal subgroup of $\mathbb{G}^1$. In the same way, $\operatorname{Ker} \mathcal{L} = \mathcal{K}$ is homogeneous ideal of $\mathfrak{g}^1$. The set is homogeneous if it is invariant under the action of dilations. Observe that an homogeneous subspace $W \subset \mathfrak{g}$ admits a decomposition in direct sum: $W = (W \cap \mathfrak{g}_1) \oplus \ldots \oplus (W \cap \mathfrak{g}_m)$.

Definition 2.3.1. A closed subgroup $\mathbb{W} \subset \mathbb{G}$ is said to be vertical if

$$\exp(\mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k) \subset \mathbb{W}.$$

Note that a vertical subgroup is automatically normal. Equipped with the natural topology, family vertical subgroups of given dimension form an open set insight all homogeneous subgroups of the same dimension.

Horizontal derivative.

Definition 2.3.2 ([Pan89]). Let $f\colon \Omega \subset \mathbb{G}^1 \to \mathbb{G}^2$ be a map defined on the open set Ω. Map f is called *horizontally differentiable* at $x \in \mathbb{G}^1$, if there is a horizontal homomorphism $L\colon \mathbb{G}^1 \to \mathbb{G}^2$ such that

$$\mathsf{d}_2\big(f(a)^{-1} \cdot f(a \cdot h), L(h)\big) = o(\mathsf{d}_1(h, e_1)) \text{ when } h \to 0.$$

If this is the case, L will be unique. So, we denote it by $D_h f(a)$ and call the *horizontal differential* of f at a.

If f is horizontally differentiable at every point $a \in \Omega$ and the differential $D_h f(a)$ depends continuously on a, then f is said to be *continuously horizontally differentiable* on Ω. The class of those maps is denoted by $C^1_h(\Omega, \mathbb{G}^2)$, or even shorter C^1_h, if the context determines without ambiguity the source and target spaces. As in the classical situation, the chain rule holds for maps in C^1_h, as well as some other arithmetic rules. Remark that $f \in C^1_h(\mathbb{G}^1, \mathbb{G}^2)$ is locally Lipschitz w.r.t d_1 and d_2.

Criterion in terms of horizontal partial derivatives. The result below (see [Vod07], for instance) is an counterpart of the classical theorem saying that the continuity of partial derivatives implies the continuous differentiability.

Theorem 2.3.3. *A map f belongs to $C^1_h(\Omega, \mathbb{G}^2)$ if and only if for each horizontal left invariant vector field $X \in H\mathbb{G}^1$ the partial derivative $Xf(a) := \frac{d}{dt}[f \circ \exp(tX)(a)]_{t=0}$ is continuous on Ω and for all $a \in \Omega$ the vector $Xf(a)$ lies in horizontal bundle: $Xf(a) \in H_{f(a)}\mathbb{G}^2$.*

It will be important to recall a generalisation of another classical result.

Theorem 2.3.4 ("Lagrange", [FS82]). *For $f \in C^1_h(\mathbb{G}, \mathbb{R})$, $a, b \in \mathbb{G}$ the following inequality is verified*

$$|f(a) - f(b) - D_h f(b)(b^{-1} \cdot a)| \le C\mathsf{d}(a, b) \max_{X \in H\mathbb{G},\, \|X\| \le 1} \|Xf(\cdot) - Xf(b)\|_{\infty,\, \bar{B}(b,r)}, \quad (2.4)$$

where the radius $r = C\mathsf{d}(a, b)$ and $C < \infty$ some constant depending only on $(\mathbb{G}, \mathsf{d})$.

In the case of non-commutative target the generalization of Theorem 2.3.4 was obtained in [Mag13, Th. 1.2]. Here we reformulate this result according to our notations.

Theorem 2.3.5 (Mean Value Inequality, [Mag13]). *Let $f \in C^1_h(\Omega, \mathbb{G}^2)$ with Ω open in $\mathbb{G}^1$. Then there exist a geometric constant $K = K(\mathbb{G}^1, \mathsf{d}_1)$ and an increasing function $c\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\mathsf{d}_2\Big(f(a)^{-1} \cdot f(b), D_h f(b)(a^{-1} \cdot b)\Big) \le KC\mathsf{d}_1(a, b) \left(\max_{X \in H\mathbb{G},\, \|X\| \le 1} \|Xf(\cdot) - Xf(b)\|_{\infty,\, \bar{B}(b,r)}\right)^{(\deg \mathbb{G}^1)^{-2}},$$

with $C = c\big(\max_{b' \in \bar{B}(b,r)} \|Df(b')\|\big)$ holds for every $a, b \in \mathbb{G}^1$ such that the ball $\bar{B}(b, r)$ of radius $r = K\mathsf{d}(a, b)$ is compactly contained in Ω.

Definition. We denote by $C_h^{1,\alpha}(\mathbb{G}^1,\mathbb{G}^2)$, $0<\alpha<1$, the space of functions $f\in C_h^1$ satisfying

$$\|D_h f(a)^{-1}\cdot D_h f(b)\| \lesssim \mathsf{d}_1(a,b)^\alpha.$$

Whitney extension theorem. This result provides the sufficient condition for an extension scalar differentiable map initially defined only on closed set to the whole space.

Theorem 2.3.6 ([VP06]). *Let $E\subset\mathbb{G}$ be a closed set. Assume that $f\colon E\to\mathbb{R}$ and $k: E\to \mathrm{Hom}_h(\mathbb{G},\mathbb{R})$ are continuous. We put*

$$R(a,b) := f(a)-f(b)-k(a)(b^{-1}\cdot a),$$

and for a compact $K\subset E$

$$\delta_K(\varepsilon) := \sup\left\{\frac{|R(a,b)|}{\mathsf{d}(a,b)} \mid a,b\in K, 0<\mathsf{d}(a,b)<\varepsilon\right\}.$$

If $\delta_K(\varepsilon)\to 0$ when $\varepsilon\to 0+$ whatever compact $K\subset F$ we take, then there exists a function $\tilde f\in C_h^1(\mathbb{G},\mathbb{R})$ such that $\tilde f_{|E}=f$ et $D_h\tilde f_{|E}=k$.

We will also need a more precise version of Whitney extension theorem in order to control the modulus of continuity of horizontal derivatives. Here we formulate it for Hölder classes.

Theorem 2.3.7 ([VP06]). *Take $K\Subset\mathbb{G}$ a compact set and fix $0<\alpha<1$. Let $f\colon K\to\mathbb{R}$ and $k: K\to\mathrm{Hom}_h(\mathbb{G},\mathbb{R})$ such that for any $a,b\in K$*

$$\|k(a)^{-1}\cdot k(b)\|\le M\mathsf{d}(a,b)^\alpha,\quad |R(a,b)|\le M\mathsf{d}(a,b)^{1+\alpha}.$$

then there exists a function $\tilde f\in C_h^{1,\alpha}(\mathbb{G},\mathbb{R})$ satisfying $\tilde f_{|K}=f$, $D_h\tilde f_{|K}=k$ and

$$\|D_h\tilde f(a)^{-1}\cdot D_h\tilde f(b)\|\lesssim M\mathsf{d}(a,b)^\alpha,\quad a,b\in\mathbb{G}.$$

3. Tangents to level sets

Some of the results of this chapter, in particular those of Sections 3.2 and 3.3, are products of joint work with F. Bigolin and were present in [BK14].

Notation. In this chapter we are going to consider either one Carnot group denoted by $(\mathbb{G}, \mathsf{d}, e)$ (corresponding to a group, metric, neutral element) either two Carnot groups denoted by $(\mathbb{G}^1, \mathsf{d}_1, e_1)$ and $(\mathbb{G}^2, \mathsf{d}_2, e_2)$. The symbol Ω will usually stay for an open set either in $\mathbb{G}$ either in $\mathbb{G}^1$.

3.1. Reifenberg flatness of level sets

3.1.1. Level sets are Reifenberg vanishing flat

Theorem 3.1.1. *Let $\mathbb{G}^1$, $\mathbb{G}^2$ be Carnot groups and let Ω be open in $\mathbb{G}^1$. Assume that $D_hF(p)$ is surjective at a point $p \in \Omega$ for a map $F \in C^1_h(\Omega, \mathbb{G}^2)$. Then there is a neighbourhood U of p in which the level set $S := F^{-1}(F(p))$ is vanishing Reifenberg flat w. r. t. kernels of horizontal differentials of F, that is, there is an increasing function $\epsilon\colon (0,\infty) \to (0,\infty)$, $\epsilon(t) \to 0+$ when $t \to 0+$, such that for every $a \in U \cap S$*

$$\operatorname{dist}_{\mathsf{d}_1}\big(B(a,r) \cap S, B(a,r) \cap a \cdot \operatorname{Ker}(D_hF(a))\big) \le \epsilon(r)r, \quad r > 0. \tag{3.1}$$

Since we are looking at objects at scale r, the function ϵ is always bounded by $\epsilon(r) \le \operatorname{diam}(B(a,r))/r \lesssim 1$.

Proof. Let us first fix a homogeneous subspace $\mathfrak{t} \subset \mathfrak{g}^1$ transversal to $\log(\operatorname{Ker} D_hF(p))$, i. e.

$$\mathfrak{t} \oplus \log(\operatorname{Ker} D_hF(p)) = \mathfrak{g}^1 \quad \text{and} \quad \delta_s(\mathfrak{t}) = \mathfrak{t},\ s > 0.$$

One can take, for instance, an orthogonal complement. Note that the choice of $\mathfrak{t}$ is not canonical. We denote $T := \exp(\mathfrak{t})$. Since D_hF is continuous and surjective at p, we can find a ball $B(p, R_0) \Subset \Omega$, $R_0 > 0$, and $\eta > 0$ such that

$$\mathsf{d}_2(D_hF(a)\langle v\rangle, e_2) \ge \eta\, \mathsf{d}_1(v, e_1), \quad v \in T, \quad a \in U. \tag{3.2}$$

Otherwise, by homogeneity, we could find a non-zero element $v \in T$ such that $v \in \operatorname{Ker} D_hF(p)$ that contradicts $\mathfrak{t} \cap \log(\operatorname{Ker} D_hF(p)) = \{0\}$.

By Theorem 2.3.5, the small “o” in the definition of horizontal differentiability is uniform for the points from any compact part of Ω. Therefore, there is an increasing function $w : \mathbb{R}_+ \to \mathbb{R}_+$, $w(r) \to 0$ for $r \to 0$, such that

$$\mathsf{d}_2(F(a)^{-1} \cdot F(b),\ D_hF(a)\langle a^{-1} \cdot b\rangle) \le w(\mathsf{d}_1(a,b))\mathsf{d}_1(a,b), \tag{3.3}$$

for all points $a, b \in \bar{B}(p, R)$. In particular, for $a, b \in F^{-1}(F(p)) \cap \bar{B}(p, R_0)$ we get that

$$\mathsf{d}_2(D_h F(a)\langle a^{-1} \cdot b\rangle, e_2) \leq w(\mathsf{d}_1(a, b))\mathsf{d}_1(a, b). \tag{3.4}$$

In Lemma 3.1.2 below we show (with a help of topological degree theory for continuous mappings) that the projection of S along T on $a \cdot \operatorname{Ker} D_h F(a)$ is (locally) surjective. So, let R_1 be as in Lemma 3.1.2 and let us put $U = B(p, R_0/3)$. To show Eq. (3.1) means for $a \in U \cap S$ to show two statements:

1. If $b \in S \cap \bar{B}(a, r)$ there is $\tilde{b} \in a \cdot \operatorname{Ker} D_h F(a) \cap \bar{B}(a, r)$ such that $\mathsf{d}_1(b, \tilde{b}) \leq \epsilon(r)r$;
2. If $\tilde{b} \in a \cdot \operatorname{Ker} D_h F(a) \cap \bar{B}(a, r)$ there is $b \in S \cap \bar{B}(a, r)$ such that $\mathsf{d}_1(b, \tilde{b}) \leq \epsilon(r)r$.

Case 1. Let $a \in S \cap U$ and $b \in S \cap B(a, r)$, $r < R_1$. We write $a^{-1} \cdot b = k \cdot v$ where $k \in \operatorname{Ker} D_h F(a)$ and $v \in T$. Such k and v are indeed unique because the map

$$\iota\colon \operatorname{Ker} D_h F(a) \times T \to \mathbb{G}^1, \quad \iota(k, v) = k \cdot v,$$

is a global diffeomorphism: since $\mathfrak{t} \oplus \log(\operatorname{Ker} D_h F(a)) = \mathfrak{g}^1$, $d\iota(e_1, 0)$ is invertible, so, by inverse function theorem, ι is a local diffeomorphism, and by homogeneity it is a global one. So, we define $\tilde{b} = a \cdot k \in a \cdot \operatorname{Ker} D_h F(a)$.

Case 2. Let $a \in S \cap U$ and $\tilde{b} = a \cdot k \in B(a, r) \cap (a \cdot \operatorname{Ker} D_h F(a))$, $r < R_1$. Using Lemma 3.1.2 we find a point $b = a \cdot k \cdot v$ with $v \in T$ such that $b \in S \cap B(a, 2R_1)$.

In both cases we can use a common estimate following from Eqs. (3.2) and (3.4)

$$\begin{aligned}
\mathsf{d}_1(b, \tilde{b}) = \mathsf{d}_1(v, e_1) &\leq \eta^{-1}\mathsf{d}_2(D_h F(a)\langle v\rangle, e_2) \\
&= \eta^{-1}\mathsf{d}_2(D_h F(a)\langle k \cdot v\rangle, e_2) = \eta^{-1}\mathsf{d}_2(D_h F(a)\langle a^{-1} \cdot b\rangle, e_2) \\
&\leq \eta^{-1} w(\mathsf{d}_1(a, b))\mathsf{d}_1(a, b).
\end{aligned}$$

By the triangle inequality,

$$\mathsf{d}_1(a, b) \leq \mathsf{d}_1(b, \tilde{b}) + \mathsf{d}_1(a, \tilde{b}) \leq \eta^{-1} w(\mathsf{d}_1(a, b))\mathsf{d}_1(a, b) + \mathsf{d}_1(a, \tilde{b}).$$

Since $\mathsf{d}_1(a, b) < R_1$ we have $w(\mathsf{d}_1(a, b)) \leq \eta/2$ and

$$\mathsf{d}_1(a, b) \leq (1 - \eta^{-1} w(\mathsf{d}_1(a, b)))\mathsf{d}_1(a, \tilde{b}) \leq 2\mathsf{d}_1(a, \tilde{b}),$$

therefore, $\mathsf{d}_1(a, b) \approx \mathsf{d}_1(a, \tilde{b}) \approx r$. Thus, following Proposition 3.1.4 we can conclude with $\epsilon(r) \lesssim (w(r)/\eta)^{1/\deg \mathbb{G}}$, $r \in (0, R_1)$. □

In the next auxiliary result we will keep the notations of the proof of Theorem 3.1.1.

Lemma 3.1.2. *Take $\varepsilon \in (0, R_0/3)$ in such a way that $|w(\varepsilon)| < \eta/2$ where w is as in Eq. (3.3). There is $R_1 \in (0, R_0/3)$ depending only on w, η, $\operatorname{Lip}(F \llcorner B(p, R_0))$ such that the following holds. Given arbitrary $a \in S \cap B(p, R_0/3)$, for any $k \in B(e_1, R_1) \cap \operatorname{Ker} D_h F(a)$ there is a point $v(k) \in T \cap B(e_1, \varepsilon)$, such that*

$$a \cdot k \cdot v(k) \in F^{-1}(F(a)).$$

Remark 3.1.3. Note that, in general, the point $v(k)$ is not necessarily unique unless T is a subalgebra. Indeed, assume that T is a subgroup and there are at least two such points $v(k), v(k)' \in T$. By Eq. (3.4),

$$\mathsf{d}_2(D_hF(a)\langle v(k)^{-1} \cdot v(k)' \rangle, e_2) \leq w(\mathsf{d}_1(v(k), v(k)'))\mathsf{d}_1(v(k), v(k)'),$$

but $v(k)^{-1} \cdot v(k)' \in T$, and, therefore, by applying Eq. (3.2),

$$\mathsf{d}_2(D_hF(a)\langle v(k)^{-1} \cdot v(k)' \rangle, e_2) \geq \eta\, \mathsf{d}_1(v(k), v(k)'),$$

we obtain a contradiction.

Proof. For $k \in B(e_1, R_0/3) \cap \operatorname{Ker} D_hF(a)$ we introduce

$$F_k(v) := F(a \cdot k \cdot v), \quad F_k : D_\varepsilon \to \mathbb{G}^2, \quad D_\varepsilon := B(e_1, \varepsilon) \cap T.$$

(We must take $\varepsilon < R_0/3$ in order that $a \cdot k \cdot D_\varepsilon \subset B(p, R_0)$.) If $v \in D_\varepsilon$, then by Eqs. (3.2) and (3.3) we get

$$\begin{aligned} \mathsf{d}_2(F_{e_1}(v), F_{e_1}(e_1)) &= \mathsf{d}_2(F(a \cdot v), F(a)) \\ &\geq \mathsf{d}_2(D_hF(a)\langle v \rangle, e_2) - w(\mathsf{d}_1(v, e_1))\mathsf{d}_1(v, e_1) \\ &> \eta\, \mathsf{d}_1(v, e_1) - \frac{\eta}{2}\mathsf{d}_1(v, e_1) = \frac{\eta}{2}\mathsf{d}_1(v, e_1). \end{aligned} \tag{3.5}$$

This implies the image of the boundary $F_{e_1}(\partial D_\varepsilon)$ does not meet $F(a)$. Let us take some $R_1 > 0$ and $k \in B(e_1, R_1) \cap \operatorname{Ker} D_hF(a)$, then for $v \in \partial D_\varepsilon$ we have (using Appendix A.5)

$$\begin{aligned} \mathsf{d}_2(F(a \cdot k \cdot v), F(a)) &\geq \mathsf{d}_2(F(a \cdot v), F(a)) - \mathsf{d}_2(F(a \cdot k \cdot v), F(a \cdot v)) \\ &\geq \frac{\eta\epsilon}{2} - \operatorname{Lip}(F \llcorner B(p, R_0))\mathsf{d}_1(k \cdot v, v) \\ &\geq \frac{\eta\epsilon}{2} - \operatorname{Lip}(F \llcorner B(p, R_0))C(\epsilon)R_1^{\frac{1}{\deg(\mathbb{G})}}. \end{aligned}$$

It is clear now that we can choose $R_1 > 0$ independent of a so that the last difference is strictly positive. This implies that $F(a) \notin F_k(\partial D_\varepsilon)$ for every $k \in B(e_1, R_1) \cap \operatorname{Ker} D_hF(a)$.

Each map $F_k \in C^0(D_\varepsilon, \mathbb{G}^2)$ is obviously homotopic to $F_{e_1} \in C^0(D_\varepsilon, \mathbb{G}^2)$ by means of F. Note that T and $\mathbb{G}^2$ are diffeomorphic due to the surjectivity of $D_hF(a)$. So, for $k \in B(e_1, R_1) \cap \operatorname{Ker} D_hF(a)$ the topological degree $\deg\big(F_k, D_\varepsilon, F(a)\big)$ (see, for instance, [Llo78]) is well defined and since it is homotopy invariant, we have

$$\deg\big(F_k, D_\varepsilon, F(a)\big) = \deg\big(F_{e_1}, D_\varepsilon, F(a)\big).$$

Observe that by Eq. (3.5), $F_{e_1}^{-1}(F(a)) \cap D_\varepsilon = \{e_1\}$. Furthermore, F_{e_1} is homotopic (by the actions of translation by $F(a)^{-1}$ and dilation δ_s, $s \to 0$) to the horizontal differential of F at a restricted on T that gives by surjectivity hypothesis

$$\deg(F_k,\ D_\varepsilon,\ F(a)) = \deg(D_hF(a),\ T,\ e_2) \in \{1, -1\}.$$

Since $\deg\big(F_k, D_\varepsilon, F(a)\big) \neq 0$, for all $k \in B(e_1, R_1) \cap \operatorname{Ker} D_hF(a)$ there exists $v(k) \in D_\varepsilon$ such that $F_k(v(k)) = F(a \cdot k \cdot v(k)) = F(a)$. □

Next two statements are technical and also needed in the proof of Theorem 3.1.1.

Proposition 3.1.4. *Let $S \subset \mathbb{G}$ be closed containing the origin e and let $W \subset \mathbb{G}$ be closed and homogeneous. Assume that for every $r > 0$,*

1. *for any $b \in S \cap B(e,r)$ there is $\tilde{b} \in W$ such that $\mathsf{d}(b,\tilde{b}) \leq r\epsilon(r)$,*
2. *for any $\tilde{b} \in W \cap B(e,r)$ there is $b \in S$ such that $\mathsf{d}(b,\tilde{b}) \leq r\epsilon(r)$.*

Then,

$$\mathrm{dist}_{\mathsf{d}}(S \cap \bar{B}(e,r), W \cap \bar{B}(e,r)) \lesssim r\epsilon(r)^{1/\deg \mathbb{G}}.$$

Proof. Imagine that in case 1. $\tilde{b}$ does not lie inside $B(e,r)$. Then $\tilde{b}$ should be replaced by $\tilde{b}'$ as in Proposition 3.1.5. Let $b \notin B(e,r)$ in case 2. This means $\tilde{b} \notin B(e, r-\epsilon(r)r)$. Then we should first consider $\tilde{b}' \in W \cap B(e, r-\epsilon(r)r)$ as in Proposition 3.1.5 (with radius $r := r - \epsilon(r)r$) for which the closest $b' \in S$ will be inside $B(e,r)$. Applying the triangle inequality allows to conclude. □

Proposition 3.1.5. *Let $W \Subset \mathbb{G}$ be a closed homogeneous set, $\tilde{b} \in W$ and $b \in B(e,r)$. Assume that $\mathsf{d}(\tilde{b},b) \leq \epsilon r$ with $0 < \epsilon \lesssim 1$. Then there is $\tilde{b}' \in W \cap B(e,r)$ such that $\mathsf{d}(\tilde{b}',b) \lesssim \epsilon^{\frac{1}{\deg \mathbb{G}}} r$.*

Proof. By rescaling, we can assume without loss of generality that $r = 1$. By the triangle inequality, $\tilde{b} \in B(e, 1+\epsilon)$, so we can take $\tilde{b}' = \delta_{1/(1+\epsilon)}(\tilde{b}) \in B(e,1)$ that also belongs to W due to homogeneity. Let us show that $\mathsf{d}(\tilde{b},\tilde{b}') \lesssim \epsilon^{1/\deg \mathbb{G}} r$. Put $\tilde{b} = \exp(\sum_i X_i)$ with $X_i \in \mathfrak{g}_i$. Then $\|X_i\|^{1/i} \lesssim 1$ and $\tilde{b}' = \exp(\sum_i X_i/(1+\epsilon)^i)$. Therefore for the Euclidean distance we have the following estimate,

$$d_{euclid}(\tilde{b},\tilde{b}') \leq \sum_i \|X_i - (1+\epsilon)^{-i} X_i\| \lesssim \epsilon \sum_i \|X_i\| \lesssim \epsilon.$$

By a basic comparison between between Euclidean and Carnot distances we get

$$\mathsf{d}(\tilde{b},\tilde{b}') \lesssim d_{euclid}(\tilde{b},\tilde{b}')^{1/\deg \mathbb{G}} \lesssim \epsilon^{1/\deg \mathbb{G}},$$

so that

$$\mathsf{d}(b,\tilde{b}') \leq \mathsf{d}(b,\tilde{b}) + \mathsf{d}(\tilde{b},\tilde{b}') \lesssim \epsilon + \epsilon^{1/\deg \mathbb{G}} \lesssim \epsilon^{1/\deg \mathbb{G}}.$$

□

3.1.2. Continuity of level sets in Hausdorff distance

In fact, Lemma 3.1.2 gives almost for free some useful consequences about the local continuity of levels sets of F equipped with the Hausdorff distance. First, we should note that F restricted to translated plans T is locally surjective.

Corollary 3.1.6. *With notations of Lemma 3.1.2, for any $a \in F^{-1}(F(p)) \cap B(p, R_0/3)$,*

$$B(F(a), \frac{\eta}{2} r) \subset F(a \cdot (T \cap B(e_1, r))), \quad 0 < r \leq \varepsilon.$$

Proof. Equation (3.5) says that $\mathsf{d}_2(F(a), F(a\cdot v)) \geq r\eta/2$ for $a \in T\cap\partial B(e_1, r)$. Therefore, the degree $\deg(F, a\cdot(T\cap B(e_1, r)), b)$ is well defined for $b \in B(F(a), r\eta/2)$ and equal to $\deg(F,(a\cdot T\cap B(e_1, r)), F(a))$ that is different from 0, so that, $b \in F(a\cdot(T\cap B(e_1, r)))$. $\square$

Using Corollary 3.1.6 we want to formulate a statement about the bi-Lipschitz continuity of level sets in the Hausdorff distance. It seems to be technically hard to give a formulation that provides an appropriate boundary condition, that is a neighbourhood [1] of a point p where $D_hF(p)$ is surjective in which level sets would be bi-Lipschitz continuous. To overcome this we introduce a local analogue of the Hausdorff distance $\mathrm{dist}_{\mathsf{d}_1}$ (see [Dav05, Sec. 34] for details).

Definition 3.1.7. Let $\Omega \subset \mathbb{G}^1$ be open set. Take $\{\Omega_m\}_{m\geq 0}$, an increasing sequence of compact sets such that $\Omega_m \subset \mathrm{int}(\Omega_{m+1})$ and $\bigcup_m \Omega_m = \Omega$. For two sets $E_1, E_2 \subset \Omega$ we define

$$\mathrm{dist}_m(E_1, E_2) := \max\Big\{ \sup_{a\in E_1\cap\Omega_m} \mathsf{d}_1(a, E_2), \sup_{b\in E_2\cap\Omega_m} \mathsf{d}_1(b, E_1)\Big\}.$$

We say $E_k \to E$ locally inside Ω if $\mathrm{dist}_m(E_k, E) \to 0$ for all $m \geq 0$.

It is continent to adapt the following convention:

- $\mathsf{d}(a, E) = +\infty$ if $E = \varnothing$,
- $\sup_{a\in E_1\cap\Omega_m} \mathsf{d}_1(a, E_2) = 0$ if $E_1 \cap \Omega_m = \varnothing$.

Proposition 3.1.8. *Let $\mathbb{G}^1$, $\mathbb{G}^2$ be Carnot groups and let Ω be open bounded set in $\mathbb{G}^1$. Take two maps $F_1, F_2 \in C^1_h(\Omega, \mathbb{G}^2)$ such that $D_hF_i(a)$, $i = 1, 2$, is surjective at every $a \in \Omega$. Let Ω_m be an exhaustion of Ω as in Definition 3.1.7. Then for $t \in F_1(\Omega)\cap F_2(\Omega)$*

$$\mathrm{dist}_m(F_1^{-1}(t), F_2^{-1}(t)) \overset{(m,F_1,F_2)}{\lesssim} \epsilon := \max_{a\in\Omega} \mathsf{d}_2(F_1(a), F_2(a)).$$

Proof. Fix a compact $\Omega_m \subset \Omega$. We need only consider $t \in F_1(\Omega_m)$ (otherwise we get 0 according to our convention). One should note that the constants $\eta, \varepsilon > 0$ in Corollary 3.1.6 can be bounded uniformly when $p \in \Omega_m$. This means that for any $p \in F_1^{-1}(t)$ there is a homogeneous subset $T_p \subset \mathbb{G}^1$ such that the image $F_1(p\cdot(T_p \cap B(e_1, r)))$ contains a ball $B(F_1(p), \eta r)$ as soon as $r \leq \varepsilon$. Recall that here we require that $p\cdot(T_p \cap B(e_1, 2\varepsilon)) \subset \Omega$. So that, if $\epsilon \leq \eta\varepsilon$ then for $r = 2\eta^{-1}\epsilon$ the intersection of $p\cdot(T_p\cap B(e_1, r))$ and $F_2^{-1}(t)$ contains some point p'. Thus,

$$\mathsf{d}_1(p, F_2^{-1}(t)) \leq \mathsf{d}_1(p, p') \leq 2\eta^{-1}\epsilon.$$

If ϵ is rather big (compared to $\eta^{-1}\mathsf{d}_1(\Omega_m, \partial\Omega)$), we can merely take any point $p' \in F_2^{-1}(t) \neq \varnothing$ and get some probably big but finite estimate (since Ω is bounded).

Our arguments are symmetric in F_1 and F_2, and the conclusion follows. $\square$

[1]This choice is much simpler when $\mathrm{Ker}\, D_hF(p)$ admits a complementary subgroup.

Corollary 3.1.9. *Let $\mathbb{G}^1$, $\mathbb{G}^2$ be Carnot groups and let Ω be bounded open set in $\mathbb{G}^1$. Assume that $D_hF(p)$ is surjective at every point of $p \in \Omega$ for a map $F \in C_h^1(\Omega, \mathbb{G}^2)$. Then, for any $t, t' \in F(\Omega)$*

$$\mathrm{Lip}(F)^{-1}\, \mathsf{d}_2(t,t') \le \mathrm{dist}_m(F^{-1}(t), F^{-1}(t')) \overset{(m,F)}{\lesssim} \mathsf{d}_2(t,t').$$

In particular, $F^{-1}(t') \to F^{-1}(t)$ locally in Ω when $t' \to t$.

Proof. The left-hand side inequality follows from Remark 3.1.11. To obtain the right-hadn side one should apply Proposition 3.1.8 to $F_1 = F$ and $F_2 = L_{t^{-1}.t'} \circ F$. □

We can also give its global version.

Proposition 3.1.10. *Let $F \in C_h^1(\mathbb{G}^1, \mathbb{G}^2)$ such that $D_hF(p)$ is surjective at every $p \in \mathbb{G}^1$. Assume furthermore the following global bounds:*

- *there is $C > \infty$ such that $\|D_hF(a)\| < C$ for $a \in \mathbb{G}^1$;*
- *there is $\eta > 0$ such that $D_hF(a)\langle B(e_1, 1)\rangle \supseteq B(e_2, \eta)$ for $a \in \mathbb{G}^1$;*
- *there is a modulus w, $w : \mathbb{R}_+ \to \mathbb{R}_+$, $w(r) \to 0$ for $r \to 0$, such that for every $a, b \in \mathbb{G}^1$ $\|D_hF(a)^{-1} \cdot D_hF(b)\| \le w(\mathsf{d}_1(a,b))$.*

Then the map $F(\mathbb{G}^1) \ni a \longrightarrow F^{-1}(a)$ is bi-Lipschitz in the Hausdorrf distance with the constants depending only on C, η, w.

In this context we find relevant to give also the following simple and general facts about the continuity of level sets.

Remark 3.1.11. If $F\colon (X, d_X) \to (Y, d_Y)$ is a Lipschitz map of metric spaces then the Hausdorff distance between level sets satisfies

$$\mathrm{Lip}(F)\ \mathrm{dist}_{d_X}(F^{-1}(y), F^{-1}(y')) \ge d_Y(y, y').$$

Proposition ([DH08])**.** *If $F\colon (X, d_X) \to (Y, d_Y)$ is a Lipschitz map of metric spaces, then the following conditions are equivalent:*

a. *The natural function $y \to F^{-1}(y)$ establishes a bi-Lipschitz equivalence between $F(X)$ and the space of fibers of F equipped with the Hausdorff distance.*

b. *There is $\mu > 0$ such that $d_X(x, F^{-1}(y)) \le \mu \cdot d_Y(F(x), y)$ for all $x \in X$ and $y \in f(X)$.*

Proposition ([DH08])**.** *Let $F\colon X \to Y$ be a surjective map of metric spaces. The following properties are equivalent:*

a. *There exists $\mu > 0$ such that for any $x \in X$ and $y \in Y$,*

$$d_X(x, F^{-1}(y)) \le \mu d_Y(F(x), y).$$

b. *There exists $\lambda > 0$ such that for any $x \in X$ and any $R > 0$,*

$$B(F(x), \lambda R) \subset F(B(x, R)).$$

3.1.3. Detecting level sets using flatness condition

Let us now present a converse of Theorem 3.1.1.

Theorem 3.1.12. *Let $S \subset \mathbb{G}$ be a connected locally closed set. Assume that to each point $a \in S$ corresponds a closed homogeneous set W_a, and W_p is a vertical subgroup of codimension N for some $p \in S$. Assume that for every relatively compact subset $S' \Subset S$ there is an increasing function $\epsilon\colon (0,\infty) \to (0,\infty)$, $\epsilon(t) \to 0+$ when $t \to 0+$, such that for any $a \in S'$*

$$\mathrm{dist}_{\mathsf{d}}\left(B(a,r) \cap S, B(a,r) \cap (a \cdot W_a)\right) \le \epsilon(r)r, \quad r > 0. \tag{3.6}$$

Then there exist an open neighbourhood U of S and a map $F \in C^1_h(U, \mathbb{R}^N)$, such that

$$S = F^{-1}(0), \quad \operatorname{Ker} D_h F(a) = W_a, \quad \forall a \in S.$$

We ought to make some comments about the statement of Theorem 3.1.12.

- We need to formulate the flatness condition only for points of compact subsets S' of S, because we deal with estimates which are uniform in $a \in S'$ and we want to avoid a possible problem near the boundary of S.
- Compare to the usual definition of Reifenberg vanishing flat sets. We use here a stronger version where "the approximated plane" W_a does not depend on scale. This automatically implies the continuity of the map $a \to W_a$, see Lemma 3.1.13.
- We have to consider only vertical subgroups as tangents because the construction of F uses Whitney's extension theorem which is known only for commutative target spaces $\mathbb{R}^N \approx \mathbb{G}/W_a$.

We postpone the proof of Theorem 3.1.12 to the next section because we need some additional ingredients based on (para)tangent cones.

Lemma 3.1.13. *Suppose that a closed set $S \subset \mathbb{G}$ is vanishing Reifenberg flat w. r. t. a family of closed homogeneous subsets $\{W_a \mid a \in S\}$ as in Eq. (3.6). Then the map $a \to W_a$ is continuous in the following sense*

$$\mathrm{dist}_{\mathsf{d}}(W_a \cap \partial B(e,1), W_b \cap \partial B(e,1)) \to 0, \quad \text{as} \quad S' \ni b \to a.$$

Proof. Let $a, b \in S$ with $d(a,b) = r > 0$ small. For $R > r > 0$ let $v \in W_a \cap \partial B(e,R)$ and take $\tilde{c} = a \cdot v \in B(a,R) \cap (a \cdot W_a)$. Using flatness at point a we can find $c \in S \cap B(a,R)$ such that $\mathsf{d}(c,\tilde{c}) \le R\epsilon(R)$. Note that $\mathsf{d}(b,c) \le R + r \le 2R$, so using flatness at point b we get a point $\tilde{c}' \in B(a,2R) \cap (b \cdot W_b)$ such that $\mathsf{d}(c,\tilde{c}') \le 2R\epsilon(2R)$. For $v' = b^{-1} \cdot \tilde{c}' \in W_b$ we obtain thanks to Appendix A.5 that

$$\begin{aligned}
3R\epsilon(2R) \ge \mathsf{d}(\tilde{c},\tilde{c}') &= \mathsf{d}(a \cdot v, b \cdot v') = \mathsf{d}(v^{-1} \cdot (b^{-1} \cdot a) \cdot v, v^{-1} \cdot v') \\
&\ge \mathsf{d}(v,v') - \mathsf{d}(v^{-1} \cdot (b^{-1} \cdot a) \cdot v, e) \ge \mathsf{d}(v,v') - C(R)r^{1/\deg(\mathbb{G})}.
\end{aligned}$$

This means that if we take $r = r(R, \epsilon(2R)) > 0$ small enough, we get $\mathsf{d}(v,v') \lesssim R\epsilon(2R)$. It is easy to see that we can arrange this in such a way that $R \to 0$ when $r(R,\epsilon(2R)) \to 0$.

(One can quantify the dependence of r on R and $\epsilon(2R)$ in order to estimate a modulus of continuity of the map $a \to W_a$ that we don't need here.) By rescaling v and v' we get that

$$\mathsf{d}(\delta_{1/\mathsf{d}(v,e)}(v), \delta_{1/\mathsf{d}(v',e)}(v')) \lesssim \epsilon(2R).$$

Of course, we could use a symmetric argument starting from $v' \in W_b \cap B(e,R)$. So, at the end we obtain that for every $R > 0$ and any two points $a, b \in S$ with $\mathsf{d}(a,b) \leq r(R, \epsilon(R))$

$$\operatorname{dist}_{\mathsf{d}}(W_a \cap \partial B(e,1), W_b \cap \partial B(e,1)) \lesssim \epsilon(2R).$$

This concludes the proof. □

3.1.4. Flat sets vs level sets.

When the target Carnot group $\mathbb{G}^2$ is not commutative, we have much less flexibility for contact maps between $\mathbb{G}^1$ and $\mathbb{G}^2$. In particular, we cannot apply Whitney extension theorem, so a compact set $S \subset \mathbb{G}^1$ satisfying a tangency condition Eq. (3.6) may not arise as a level set of a horizontally differentiable map $F\colon \mathbb{G}^1 \to \mathbb{G}^2$ with surjective differential.

For instance, we show in Lemma A.2.1 that there is an example of rigid Carnot groups $\mathbb{G}^1$ and $\mathbb{G}^2$ such that any C^1_h-map between them with surjective horizontal differential is up to translation a group homomorphism. That emphases a big gap that can appear between sets with tangency condition and actual level sets. Note that locally any smooth curve tangent to centre$(\mathbb{G}^1)$ in Lemma A.2.1 satisfies the corresponding tangency condition.

The situation is not even so clear for a rather large [War05] set of the contact maps between jet spaces. For instance, a simple question can be: is there a C^1_h-map from the Engel group $\mathbb{E}^4$ (see Section 2.2.2) to the first Heisenberg group $\mathbb{H}^1$ with surjective differential admitting an "irregular" level set that, let say,

1. is merely different from its tangent;
2. is not bi-Lipschitz equivalent to its tangent.

On the example of $\mathbb{E}^4$ we are going to show that there is a large variety of sets satisfying the tangency condition Eq. (3.6). We shall use the next technical result.

Proposition 3.1.14. *Take $Z \in \operatorname{centre}(\mathfrak{g})$ be a homogeneous vector field from the centre of $\mathfrak{g}$. Let $\Gamma\colon [0,1] \to \mathbb{G}$ be Hölder continuous with exponent $\alpha > 1/\deg Z$. Then there is a large enough constant $K = K(\|\Gamma\|_{\mathrm{Hol}^\alpha}) > 0$ such that the image S of the modified curve*

$$t \to \Gamma(t) \cdot \exp(KtZ)$$

satisfies Eq. (3.6) with $W_a = \exp(Z)$ for any $a \in S$.

This result generalizes Proposition 5.5.8 whose proof is very similar and we will drop it. On the contrary, let us describe a possible construction of such curves Γ for $\alpha \in (1/2, 1]$ that are tangent to $Z \in \mathfrak{g}$, $\deg Z = 3$, in $\mathbb{E}^4$

Take any curve $[0,1] \ni t \to \gamma(t) = (v(t), x(t)) \in \mathbb{R}^2$ of Hölder exponent α. We can apply an extension theorem (for instance, [Lyo98, Th. 2.1.1], [LQ02, Th. 3.1.2], with $p = 1/\alpha < 2$ and $n = 1$) to γ in order to construct a "lifted" curve $\Gamma' \colon [0,1] \to (\mathbb{G}_3(\mathbb{R}^2), \mathsf{d})$ with values in the free Carnot group of step 3 and rank 2 that is Hölder continuous with the same exponent α. The curve Γ' is a lift of γ in the sense that $\pi(\Gamma') = \gamma$ where $\pi : \mathbb{G}_3(\mathbb{R}^2) \to \mathbb{R}^2$ is the canonical projection. Note also that the curve Γ' is unique with those properties.

By the basic property of free Carnot groups, we can take a quotient of $\mathbb{G}_3(\mathbb{R}^2)$ by a well chosen normal subgroup (containing all extra commutators) to obtain $\mathbb{E}^4$. This gives a rise to a projection $\mathrm{Pr} : \mathbb{G}_3(\mathbb{R}^2) \to \mathbb{E}^4$. This projection is always Lipschitz and commutes with projections on horizontal planes $\mathrm{Pr} \circ \pi = \pi' \circ \mathrm{Pr}$, where $\pi'(v,x,y,z) = (v,x)$ in coordinates on $\mathbb{E}^4$. So, we should take $\Gamma = \mathrm{Pr}(\Gamma') \in \mathrm{Hol}^\alpha([0,1], \mathbb{E}^4)$ for which $\pi'(\Gamma) = \gamma$.

The technique presented above is quite general and can be used to construct lifted "horizontal" curves in an arbitrary Carnot group.

But there is also a way to construct this "horizontal lift" directly in $\mathbb{E}^4$ without passing via $\mathbb{G}_3(\mathbb{R}^2)$. Having (v,x) we should solve step by step the contact equations

$$\begin{aligned} \frac{1}{2} x\, dv - \frac{1}{2} v\, dx + dy = 0, \\ \Big(\frac{1}{2} y - \frac{1}{6} xv\Big)\, dv + \frac{1}{6} v^2\, dx - \frac{1}{2} v\, dy + dz = 0. \end{aligned}$$

(in rough sense) to find the y and z components. Concretely, first we put $y(t) = 1/2 \int_0^t v\, dx - x\, dv$ as an integral in Stieltjes sense, Theorem A.1.3. Now we note that y is also a Hölder function with the same exponent α (see Remark 5.5.10). Obviously, the functions xv and v^2 are also α-Hölder. Therefore, we can define z again as a Stieltjes integral. Let us check that the curve $\Gamma = (v,x,y,z)$ obtained in this way is α-Hölder continuous in Carnot distance, that is

$$\mathsf{d}(\Gamma(t), \Gamma(s)) \lesssim |t-s|^\alpha, \quad t, s \in [0,1].$$

Let $t, s \in [0,1]$, then by hypothesis we already have this bound for the x and v components of $\Gamma(s)^{-1} \cdot \Gamma(t)$. For the y component, it immediately follows from the estimate in Theorem A.1.3. So, we only need to deal with the z component.

Since Γ is uniquely defined by left-invariant differential equations, we can perform a left-translation and assume without loss of generality that $\Gamma(s) = e$. Otherwise we would have a non-zero boundary term in the computation below. For the sake of simplicity we also perform a translation in time and assume that $s = 0$ and $t > 0$. Finally, we need to prove that

$$|z(t)| = \Big| \int_0^t \Big(\frac{1}{2} y - \frac{1}{6} xv\Big)\, dv + \frac{1}{6} v^2\, dx - \frac{1}{2} v\, dy \Big| \lesssim t^{3\alpha},$$

if $v(0) = x(0) = y(0) = 0$. By Young's estimate in Theorem A.1.3, we have

$$\Big| \int_0^t xv\, dv \Big| \lesssim t^{2\alpha} \|v\|_{\mathrm{Hol}^\alpha} \|xv\|_{\mathrm{Hol}^\alpha} \lesssim t^{2\alpha} \|v\|_{\mathrm{Hol}^\alpha} (\|x\|_\infty \|v\|_{\mathrm{Hol}^\alpha} + \|v\|_\infty \|x\|_{\mathrm{Hol}^\alpha}).$$

But the sup-norms of v and x on the interval $[0,t]$ are bounded by

$$\|v\|_\infty \lesssim t^\alpha \|v\|_{\mathrm{Hol}^\alpha}, \quad \|x\|_\infty \lesssim t^\alpha \|x\|_{\mathrm{Hol}^\alpha}.$$

Hence,

$$|\int_0^t xv\, dv| \lesssim t^{3\alpha}\|v\|_{\mathrm{Hol}^\alpha}(\|v\|^2_{\mathrm{Hol}^\alpha} + \|x\|^2_{\mathrm{Hol}^\alpha}).$$

The same argument applies to $\int_0^t v^2\, dx$. To get a good control on $\int_0^t y\, dv$ or $\int_0^t v\, dy$ is enough to bound the Hölder norm of y on $[0,t]$, that can be done as follows: if $t_1, t_2 \in [0,t]$, then

$$\begin{aligned}
|y(t_1) - y(t_2)| &\le |y(t_1) - y(t_2) - \frac{1}{2}(x(t_1)v(t_2) - x(t_2)v(t_1))| \\
&\quad + \frac{1}{2}|x(t_1)v(t_2) - x(t_2)v(t_1)| \\
&\lesssim \|v\|_{\mathrm{Hol}^\alpha}\|x\|_{\mathrm{Hol}^\alpha}|t_1 - t_2|^{2\alpha} + (\|x\|_\infty\|v\|_{\mathrm{Hol}^\alpha} + \|v\|_\infty\|x\|_{\mathrm{Hol}^\alpha})|t_1 - t_2|^\alpha \\
&\lesssim t^\alpha |t_1 - t_2|^\alpha \|v\|_{\mathrm{Hol}^\alpha}\|x\|_{\mathrm{Hol}^\alpha}.
\end{aligned}$$

3.2. Tangency and Paratangency in Carnot groups

Let us first give the definition of tangent and paratangent sets in Carnot group $(\mathbb{G}, \mathsf{d}, \delta)$.

Definition 3.2.1. Let $S \subset \mathbb{G}$ and $a \in \mathbb{G}$.

- $v \in \mathbb{G}$ belongs to $\mathrm{Tan}^+_{\mathbb{G}}(S,a)$ (an upper tangent cone to S in a) iff

$$\begin{aligned}
&\exists \{t_m\}_m \subset \mathbb{R}_+, \quad \lim_m t_m = 0, \quad \exists \{a_m\}_m \subset S \text{ such that} \\
&\lim_m a_m = a, \quad \lim_m \delta_{1/t_m}(a^{-1} \cdot a_m) = v.
\end{aligned} \tag{3.7}$$

- $v \in \mathbb{G}$ belongs to $\mathrm{Tan}^-_{\mathbb{G}}(S,a)$ (a lower tangent vector to S in a) iff

$$\begin{aligned}
&\forall \{t_m\}_m \subset \mathbb{R}_+, \quad \lim_m t_m = 0, \quad \exists \{a_m\}_m \subset S \text{ such that} \\
&\lim_m a_m = a, \quad \lim_m \delta_{1/t_m}(a^{-1} \cdot a_m) = v.
\end{aligned} \tag{3.8}$$

- $v \in \mathbb{G}$ belongs to $\mathrm{pTan}^+_{\mathbb{G}}(S,a)$ (an upper paratangent vector to S in a) iff

$$\begin{aligned}
&\exists \{t_m\}_m \subset \mathbb{R}_+, \quad \lim_m t_m = 0, \quad \exists \{a_m\}_m, \{b_m\}_m \subset S \text{ such that} \\
&\lim_m a_m = a, \quad \lim_m \delta_{1/t_m}(b_m^{-1} \cdot a_m) = v.
\end{aligned} \tag{3.9}$$

- $v \in \mathbb{G}$ belongs to $\mathrm{pTan}^-_{\mathbb{G}}(S,a)$ (a lower paratangent vector to S in a) iff

$$\begin{aligned}
&\forall \{t_m\}_m \subset \mathbb{R}_+, \quad \lim_m t_m = 0, \quad \forall \{a_m\}_m \subset S, \quad \lim_m a_m = a, \\
&\exists \{b_m\}_m \subset S \text{ such that } \lim_m \delta_{1/t_m}(b_m^{-1} \cdot a_m) = v.
\end{aligned} \tag{3.10}$$

Remark. The following inclusions follow from definitions

$$\mathrm{pTan}^-_{\mathbb{G}}(S,a) \subseteq \mathrm{Tan}^-_{\mathbb{G}}(S,a) \subseteq \mathrm{Tan}^+_{\mathbb{G}}(S,a) \subseteq \mathrm{pTan}^+_{\mathbb{G}}(S,a).$$

In order to define some properties of tangent and paratangent cones, let us introduce the so-called Kuratowski limits of sets. Let A_a be a subset of $\mathbb{G}$ indexed by points $a \in S \subset \mathbb{G}$. The *lower limit* $\operatorname*{Li}_{S\ni a\to a_0} A_a$ and *upper limit* $\operatorname*{Ls}_{S\ni a\to a_0} A_a$ are defined by

$$v \in \operatorname*{Li}_{S\ni a\to a_0} A_a \iff \begin{cases} \forall \{a_m\}_m \subset S \text{ with } a_m \to a_0, \exists\, \{b_m\}_m \text{ with} \\ b_m \in A_{a_m} \text{ eventually } \text{ such that } \lim_m b_m = v. \end{cases}$$

$$v \in \operatorname*{Ls}_{S\ni a\to a_0} A_a \iff \begin{cases} \exists \{a_m\}_m \subset S \text{ with } a_m \to a_0, \exists\, \{b_m\}_m \text{ with} \\ b_m \in A_{a_m} \text{ eventually } \text{ such that } \lim_m b_m = v. \end{cases}$$

In a metric space, Kuratowski convergence is weaker than convergence in Hausdorff distance. These notions of convergence coincide when restricted to compact sets.

Remark 3.2.2. Notice that, by definition, $\mathrm{Li}_{a\to a_0} A_a \subset A_{a_0}$ for any family of subsets A_a.

Remark 3.2.3. By definition, $e \subset \mathrm{pTan}^-_{\mathbb{G}}(S,a)$. Note also that $\mathrm{Tan}^-_{\mathbb{G}}(S,a) = \{e\}$ if and only if a is isolated point of S.

Let us give some simple properties of (para)tangent cones. First one justifies the name "cone".

Proposition 3.2.4. *Tangent and paratangent cones are closed and homogeneous sets containing the origin e.*

Proof. Homogeneity follows straightforward from the definition. To show closedness one should use Cantor's standard diagonal argument. □

Proposition 3.2.5. *The lower paratangent cone* $\mathrm{pTan}^-_{\mathbb{G}}(S,a)$ *is $\mathbb{G}$-convex, i. e. satisfies*

$$v_1, v_2 \in \mathrm{pTan}^-_{\mathbb{G}}(S,a) \Longrightarrow v_1 \cdot \delta_t(v_1^{-1} \cdot v_2) \in \mathrm{pTan}^-_{\mathbb{G}}(S,a), \quad t \in [0,1].$$

Proof. Let us prove that $v_1, v_2 \in \mathrm{pTan}^-_{\mathbb{G}}(S,a) \Longrightarrow v_1 \cdot v_2 \in \mathrm{pTan}^-_{\mathbb{G}}(S,a)$. Fix two sequences $\{a_m\}_m \subset S$, $\{t_m\}_m \subset \mathbb{R}_+$ such that $\lim_m \mathsf{d}(a_0, a_m) = 0$ and $\lim_m t_m = 0$. Since $v_1 \in \mathrm{pTan}^-_{\mathbb{G}}(S,a)$, there exists $\{b_m\}_m \subset S$ such that

$$\lim_m \delta_{1/t_m}(b_m^{-1} \cdot a_m) = v_1.$$

Since $\lim_m \mathsf{d}(a_0, b_m) = 0$ and $v_2 \in \mathrm{pTan}^-_{\mathbb{G}}(S,a)$, there exists $\{c_m\}_m \subset S$ such that

$$\lim_m \delta_{1/t_m}(c_m^{-1} \cdot b_m) = v_2. \tag{3.11}$$

We can conclude that $v_1 \cdot v_2 \in \mathrm{pTan}^-_{\mathbb{G}}(S,a)$, because for all $\{a_m\}_m \subset S$, $\{t_m\}_m \subset \mathbb{R}_+$ as above we can find a sequence $\{c_m\}_m \subset S$, chosen as in Eq. (3.11), such that

$$\lim_m \delta_{1/t_m}(c_m^{-1} \cdot a_m) = \lim_m \delta_{1/t_m}\Big((c_m^{-1} \cdot b_m) \cdot (b_m^{-1} \cdot a_m)\Big) = v_2 \cdot v_1.$$

Since $\mathrm{pTan}^-_{\mathbb{G}}(S,a)$ is homogeneous, we can conclude by $v_1 \cdot \delta_t(v_1^{-1} \cdot v_2) = \delta_{1-t}(v_1) \cdot \delta_t(v_2) \in \mathrm{pTan}^-_{\mathbb{G}}(S,a)$. □

Proposition 3.2.6. *The upper paratangent cone* $\mathrm{pTan}^+_{\mathbb{G}}(S,a)$ *is bilateral, i. e. satisfies*

$$v \in \mathrm{pTan}^+_{\mathbb{G}}(S,a) \Longrightarrow v^{-1} \in \mathrm{pTan}^+_{\mathbb{G}}(S,a).$$

Proof. Let $v \in \mathrm{pTan}^+_{\mathbb{G}}(S,a)$. By definition, there exist three sequences $\{a_m\}_m, \{b_m\}_m \subset S$, $\{t_m\}_m \subset \mathbb{R}_+$ such that

$$\lim_m \mathsf{d}(a,a_m) = 0, \quad \lim_m t_m = 0, \quad \lim_m \delta_{1/t_m}(b_m^{-1} \cdot a_m) = v.$$

Notice that in this case $\lim_m \delta_{1/t_m}(a_m^{-1} \cdot b_m) = v^{-1}$, therefore $v^{-1} \in \mathrm{pTan}^+_{\mathbb{G}}(S,a)$. □

Remark. Observe that if $\mathrm{pTan}^+_{\mathbb{G}}(S,a) = \mathrm{pTan}^-_{\mathbb{G}}(S,a)$, i. e. so all four cones coincide at some point $a \in S$ then by Propositions 3.2.5 and 3.2.6 they will be homogeneous subgroups.

We present now a characterization of horizontal differentiability in a Carnot group $\mathbb{G}$.

Proposition 3.2.7. *Let* $\Omega \subset \mathbb{G}^1$ *be an open set. Let* $F\colon \Omega \to \mathbb{G}^2$ *be a map between Carnot groups. The following properties are equivalent:*

i. F *is horizontally differentiable at* $a \in \Omega$ *and* $L = D_h F(a)$.

ii. *There exists a homogeneous homomorphism* $L\colon \mathbb{G}^1 \to \mathbb{G}^2$ *such that*

$$\lim_m \delta_{1/t_m}\big(F(a)^{-1} \cdot F(a_m)\big) = L\langle v\rangle$$

for each $v \in \mathbb{G}$ *and for every* $\{t_m\}_m \subset \mathbb{R}_+$ *and* $\{a_m\}_m \subset \Omega$ *as in Eq. (3.7).*

Proof. Both implications are easy consequences of the definition of horizontal differentiability. □

We also give its paratangent version.

Proposition 3.2.8. *Let* $\Omega \subset \mathbb{G}^1$ *be an open set. Let* $F\colon \Omega \to \mathbb{G}^2$ *be a map between Carnot groups. The following properties are equivalent:*

i. F *is* <u>*uniformly*</u> *horizontally differentiable at* $a \in \Omega$ *with* $L = D_h F(a)$*, meaning that there is a horizontal homomorphism* $L\colon \mathbb{G}^1 \to \mathbb{G}^2$ *such that*

$$\lim_{c,b\to a} \frac{\mathsf{d}_2(F(b)^{-1} \cdot F(c), L\langle b^{-1} \cdot c\rangle)}{\mathsf{d}_1(c,b)} = 0.$$

ii. *There exists a homogeneous homomorphism* $L\colon \mathbb{G}^1 \to \mathbb{G}^2$ *such that*

$$\lim_m \delta_{1/t_m}\big(F(b_m)^{-1} \cdot F(a_m)\big) = L\langle v\rangle$$

for each $v \in \mathbb{G}$ *and for every* $\{t_m\}_m \subset \mathbb{R}_+$*,* $\{b_m\}_m \subset \Omega$ *and* $\{a_m\}_m \subset \Omega$ *as in Eq. (3.9).*

At the end of this section we present some result about the behaviour of tangents w. r. t. limits.

Lemma 3.2.9. *Let $S \subset \mathbb{G}$ be a closed connected set. Then*

$$\mathrm{pTan}^-_{\mathbb{G}}(S, p_0) \subset \operatorname*{Li}_{S \ni p \to p_0} \mathrm{Tan}^+_{\mathbb{G}}(S, p).$$

Proof. We adapt here a classical argument from [AF65]. Let us consider $v \in \mathrm{pTan}^-_{\mathbb{G}}(S, p_0)$. Fix any sequence $\{p_m\}_{m \geq 0} \subset S$ converging to p. It can be shown (by a contradiction argument, for instance) that $v \in \mathrm{pTan}^-_{\mathbb{G}}(S, p_0)$ implies that for every $\epsilon > 0$ there exist $N \in \mathbb{N}$ and $\beta > 0$ such that for any $h \in (0, \beta]$ and any $m \geq N$ we can find $p^h_m \in S$ such that

$$\mathsf{d}(p_m \cdot v, p^h_m) \leq h\epsilon.$$

By compactness, for any $m \geq N$ there is a limit point v_m of the sequence $\{p_m^{-1} \cdot p^h_m\}_{0<h\leq\beta}$ when $h \to 0$. By definition, v_m belongs to $\mathrm{Tan}^+_{\mathbb{G}}(S, p_m)$ and $\mathsf{d}(v_m, v) < \epsilon$. Since $\epsilon > 0$ is arbitrary, we obtain the statement of Lemma. □

Remark 3.2.10. By Remark 3.2.2, $\operatorname*{Li}_{S \ni p \to p_0} \mathrm{Tan}^+_{\mathbb{G}}(S, p) \subset \mathrm{Tan}^+_{\mathbb{G}}(S, p_0)$ for $S \subset \mathbb{G}$.

By a direct application of the Cantor diagonal method we achieve the following

Lemma 3.2.11. *Let $S \subset \mathbb{G}$ be a closed connected set. Then*

$$\operatorname*{Ls}_{S \ni p \to p_0} \mathrm{pTan}^+_{\mathbb{G}}(S, p) \subset \mathrm{pTan}^+_{\mathbb{G}}(S, p_0).$$

Lemma 3.2.12. *Let $E, E' \subset \mathbb{G}$ be two closed subsets with $E \subset E'$. Assume that at some point $p \in E$,*

$$\mathrm{pTan}^-_{\mathbb{G}}(p, E) = \mathrm{pTan}^+_{\mathbb{G}}(p, E').$$

Then E coincides with E' in some neighbourhood of p.

Proof. By contradiction, assume that there is a sequence $\{p_n\} \subset E' \setminus E$ such that $p_n \to p$. Define $b_n \in E$ as a point such that $\mathsf{d}(b_n, p_n) = \min_{b \in E} \mathsf{d}(b, p_n)$. Up to extracting a subsequence, $\delta_{\mathsf{d}(p_n, b_n)^{-1}}(b_n^{-1} \cdot p_n)$ converges (by definition) to some $v \in \mathrm{pTan}^+(p, E')$. By hypothesis, $v \in \mathrm{pTan}^-_{\mathbb{G}}(p, E)$, which means that we can find a sequence $E \ni \tilde{p}_n \to p$ such that

$$\delta_{\mathsf{d}(b_n, p_n)^{-1}}(b_n^{-1} \cdot \tilde{p}_n) = v.$$

Thus, $\delta_{\mathsf{d}(b_n, p_n)^{-1}}(\tilde{p}_n^{-1} \cdot p_n) \to v \cdot v^{-1} = e$, or, equivalently, $\mathsf{d}(\tilde{p}_n^{-1}, p_n) = o(\mathsf{d}(b_n, p_n))$. This contradicts the minimality in the definition of b_n. □

3.3. Four cones Theorem

The goal of this section is to provide a link between (para)tangent cones and Reifenberg flatness for sets in Carnot groups.

Theorem 3.3.1. *Let $S \subset \mathbb{G}$ is closed set and $a \in S$. Two following conditions are equivalent:*

I. *There is a closed homogeneous set W such that*

$$r^{-1}\operatorname{dist}_{\mathsf{d}}\big(B(a,r)\cap S, B(a,r)\cap(a\cdot W)\big)\to 0, \quad r\to 0. \tag{3.12}$$

II. *Two tangent cones to S coincide at a*

$$\mathrm{Tan}^-_{\mathbb{G}}(S,a)=\mathrm{Tan}^+_{\mathbb{G}}(S,a).$$

If these conditions are fulfilled then

$$W=\mathrm{Tan}^-_{\mathbb{G}}(S,a)=\mathrm{Tan}^+_{\mathbb{G}}(S,a).$$

Proof. There will be two parts.

I $\Rightarrow$ II. Let us first prove that $\mathrm{Tan}^+_{\mathbb{G}}(S,a)\subset W$. Take $v\in\mathrm{Tan}^+_{\mathbb{G}}(S,a)$. By definition there are $\{a_m\}_m\subset S$, $\{t_m\}_m\subset\mathbb{R}_+$ such that

$$\lim_m \mathsf{d}(a,a_m)=0, \quad \lim_m t_m=0, \quad \lim_m \delta_{1/t_m}(a^{-1}\cdot a_m)=v.$$

We can find a sequence $b_m\in(a\cdot W)$ such that $\lim_m \mathsf{d}(b_m,a_m)/\mathsf{d}(a,a_m)=0$. Since $\mathsf{d}(v,e)<\infty$, the ratio $\mathsf{d}(a,a_m)/t_m$ stays bounded, so that

$$\lim_m\frac{\mathsf{d}(b_m,a_m)}{t_m}=\lim_m\frac{\mathsf{d}(b_m,a_m)}{\mathsf{d}(a,a_m)}\frac{\mathsf{d}(a,a_m)}{t_m}=0.$$

Hence,

$$v=\lim_m\delta_{1/t_m}(a^{-1}\cdot a_m)=\lim_m\delta_{1/t_m}(a^{-1}\cdot b_m\cdot b_m^{-1}\cdot a_m)=\lim_m\delta_{1/t_m}(a^{-1}\cdot b_m)\in W$$

because W is closed and homogeneous.

Now, we prove that $W\subset\mathrm{Tan}^-_{\mathbb{G}}(S,p)$. Let $v\in W$ and fix an arbitrary sequence $\{t_m\}_m\subset\mathbb{R}_+$ going to 0. Put $b_m=a\cdot\delta_{t_m}(v)\in a\cdot W$ and define $a_m\in S$ as a point in $S\cap B(a,r)$, $r=\mathsf{d}(a,b_m)$ closest to b_m. By hypothesis, $\lim_m\mathsf{d}(b_m,a_m)/\mathsf{d}(a,b_m)=\lim_m\mathsf{d}(b_m,a_m)/(t_m\mathsf{d}(v,e))=0$. Then we conclude by

$$\lim_m\delta_{1/t_m}(a\cdot a_m)=\lim_m\delta_{1/t_m}(a\cdot b_m)=v.$$

II $\Rightarrow$ I. Let $W=\mathrm{Tan}^-_{\mathbb{G}}(S,a)$. Assume that $W=\mathrm{Tan}^+_{\mathbb{G}}(S,a)$. Then W is a closed homogeneous set. Suppose by contradiction that there are $\eta>0$ and a positive sequence $r_m\to 0$ such that

$$r_m^{-1}\operatorname{dist}_{\mathsf{d}}\big(B(a,r_m)\cap S, B(a,r_m)\cap(a\cdot W)\big)\geq\eta.$$

Up to extracting a subsequence, there are two non-exclusive possibilities:

1. there exists $w_m\in W\cap B(e,r_m)$ such that $\mathsf{d}(a\cdot w_m, S\cap B(a,r_m))\geq\eta r_m$;

2. there exists $a_m \in S \cap B(a, r_m)$ such that $\mathsf{d}(a_m, (a \cdot W) \cap B(a, r_m)) \geq \eta r_m$.

In case 1., by compactness we may assume without loss of generality that $\delta_{1/r_m}(w_m) \to w \in W$. Since $w \in \mathrm{Tan}^-_{\mathbb{G}}(S, a)$, for a sequence $r_m \to 0$ we can find $\{a_m \in S\}$ such that $a_m \to a$ and

$$\mathsf{d}(\delta_{1/r_m}(a^{-1} \cdot a_m), w) = r_m^{-1}\mathsf{d}(a_m, a \cdot \delta_{r_m}(w)) \to 0, \quad m \to \infty.$$

If $a_m \notin \bar{B}(a, r_m)$, one should use an additional argument as in Proposition 3.1.4. We can finally get a contradiction by

$$\begin{aligned} 0 < \eta &\leq r_m^{-1}\mathsf{d}(a \cdot w_m, S \cap B(a, r_m)) \leq r_m^{-1}\mathsf{d}(a \cdot w_m, a_m) \\ &\leq r_m^{-1}(\mathsf{d}(a \cdot w_m, a \cdot \delta_{r_m}(w)) + \mathsf{d}(a \cdot \delta_{r_m}(w), a_m)) \to 0. \end{aligned}$$

In case 2., we may assume by extracting a subsequence that $\delta_{1/r_m}(a^{-1} \cdot a_m) \to w$. By definition, $w \in \mathrm{Tan}^+_{\mathbb{G}}(S, a)$ and we put $w_m = \delta_{r_m}(w) \in W$. The rest of the argument repeats case 1. □

Here is a trivial consequence of Theorem 3.3.1.

Corollary. *If there is a closed homogeneous set W that satisfies Eq. (3.12) then such a set is unique.*

Theorem 3.3.2. *Let $S \subset \mathbb{G}$ is closed set and $U \subset \mathbb{G}$ is an open set. The two following conditions are equivalent:*

I. *For any $U' \Subset U$ there are a family of closed homogeneous sets $\{W_a \subset \mathbb{G}\}_{a \in S}$ and a function $\epsilon(r) \to 0$, $r \to 0+$, such that*

$$\mathrm{dist}_{\mathsf{d}}\left(B(a, r) \cap S, B(a, r) \cap (a \cdot W_a)\right) \leq \epsilon(r) r, \tag{3.13}$$

for all $a \in S \cap U'$.

II. *The two paratangent cones to S coincide at every point $a \in S \cap U$*

$$\mathrm{pTan}^-_{\mathbb{G}}(S, a) = \mathrm{pTan}^+_{\mathbb{G}}(S, a). \tag{3.14}$$

If those conditions are fulfilled then

$$W_a = \mathrm{pTan}^-_{\mathbb{G}}(S, a) = \mathrm{pTan}^+_{\mathbb{G}}(S, a)$$

is a homogeneous subgroup for $a \in S \cap U$ and the map $a \to W_a$ is continuous on $S \cap U$.

Proof. There will be two parts.

I $\Rightarrow$ II. Let $a \in S \cap U$. We are going to prove first that any $v \in \mathrm{pTan}^+_{\mathbb{G}}(S,a)$ belongs to W_a. By definition, there are $\{a_m\}_m \subset S$, $\{b_m\}_m \subset S$, $\{t_m\}_m \subset \mathbb{R}_+$ such that Eq. (3.9) holds. Since U is open, we can assume that the sequences $\{a_m\}_m$ and $\{b_m\}_m$ take their values inside $S \cap U'$ with $U' \Subset U$. Using Eq. (3.13), we can find $v_m \in W_{a_m} \cap B(e, r_m)$ such that $\mathsf{d}(b_m, a_m \cdot v_m) \leq \epsilon(r_m) r_m$ with $r_m = \mathsf{d}(a_m, b_m)$. By homogeneity of W_{a_m}, $\delta_{1/t_m}(v_m) \in W_{a_m}$ and it is easy to check that $\mathsf{d}(v_m, v) \lesssim \epsilon(r_m) \to 0$ as $m \to \infty$. We know from Lemma 3.1.13 that map $p \to W_p$ is continuous on $S \cap U'$. That implies that $v = \lim_m v_m \in W_a$.

Now, let us show that any $v \in W_a$ belongs to $\mathrm{pTan}^-_{\mathbb{G}}(S,a)$. So, fix arbitrary sequences $\{a_m\}_m \subset S$, $\{t_m\}_m \subset \mathbb{R}_+$ such that $\lim_m a_m = a$ and $\lim_m t_m = 0$. Pick an element $v_m \in W_{a_m}$ which is closest to v. By Lemma 3.1.13, $\lim_m \mathsf{d}(v_m, v) = 0$. By Eq. (3.13), used in reverse direction, we can define a sequence $\{b_m\}_m \subset S$, $b_m \to a$, in such a way that $\mathsf{d}(b_m, a_m \cdot \delta_{t_m}(v_m)) \leq \epsilon(r_m) r_m$ with $r_m = t_m \mathsf{d}(v_m, e)$. Since $\mathsf{d}(v,e) < \infty$, $r_m \approx t_m$ and

$$\lim_m \mathsf{d}(\delta_{1/t_m}(a_m^{-1} \cdot b_m), v) \leq \lim_m \mathsf{d}(v_m, v) + \lim_m \mathsf{d}(\delta_{1/t_m}(a_m^{-1} \cdot b_m), v_m) \leq \lim_m \epsilon(r_m) = 0,$$

so that $v \in \mathrm{pTan}^-_{\mathbb{G}}(S,a)$.

II $\Rightarrow$ I. We want to show that $W_a := \mathrm{pTan}^-_{\mathbb{G}}(S,a) = \mathrm{pTan}^+_{\mathbb{G}}(S,a)$ satisfies Eq. (3.13). Observe that paratangent cones coincidence also guarantees the continuity of the map $p \to W_p$ on $S \cap U$ thanks to Lemmas 3.2.9 and 3.2.11 (in the same topology as in Lemma 3.1.13).

Let us argue by contradiction. Assume that there are a sequence $\{a_m\}_m \subset S \cap U'$, $U' \Subset U$, $a_m \to a \in U$, a sequence $\{r_m\}_m \subset \mathbb{R}_+$, $r_m \to 0$ and $\eta > 0$ such that one of two cases is realized:

1. there exists $w_m \in W_{a_m} \cap B(e, r_m)$ such that $\mathsf{d}(a_m \cdot w_m, S \cap B(a, r_m)) \geq \eta r_m$;
2. there exists $b_m \in S \cap B(a_m, r_m)$ such that $\mathsf{d}(b_m, (a_m \cdot W_{a_m}) \cap B(a_m, r_m)) \geq \eta r_m$.

In case 1., by compactness we may assume without loss of generality that

$$W_{a_m} \cap \partial B(e,1) \ni \delta_{1/r_m}(w_m) \to w \in W_a \cap \partial B(e,1).$$

Since $w \in \mathrm{pTan}^-_{\mathbb{G}}(S,a)$, we can find $\{b_m \in S\}$ such that $b_m \to a$ and

$$\mathsf{d}(\delta_{1/r_m}(a_m^{-1} \cdot b_m), w) = r_m^{-1} \mathsf{d}(b_m, a_m \cdot \delta_{r_m}(w)) \to 0, \quad m \to \infty.$$

If $b_m \notin \bar{B}(a_m, r_m)$, one should modify it as in Proposition 3.1.4. We can get a contradiction by

$$\begin{aligned} 0 < \eta \leq r_m^{-1} \mathsf{d}(a_m \cdot w_m, S \cap B(a_m, r_m)) &\leq r_m^{-1} \mathsf{d}(a_m \cdot w_m, b_m) \\ &\leq r_m^{-1} (\mathsf{d}(a_m \cdot w_m, a_m \cdot \delta_{r_m}(w)) + \mathsf{d}(a_m \cdot \delta_{r_m}(w), b_m)) \to 0. \end{aligned}$$

In case 2., we may assume by extracting a subsequence that $\delta_{1/r_m}(a_m^{-1} \cdot b_m) \to w \in W_a$. By definition, $w \in \mathrm{pTan}^+_{\mathbb{G}}(S,a)$ and we put $w_m = \delta_{r_m}(w) \in W_a$. The rest of the argument repeats case 1.

At the end we note that W_a has to be a subgroup. This follows from Propositions 3.2.5 and 3.2.6. □

Now we have all necessary element to prove of Theorem 3.1.12.

Proof of Theorem 3.1.12. Lemma 3.1.13 provides the continuity of the map $S \ni a \to W_a$. It turns out that each W_a, $a \in S$, is a subgroup. This is a consequence of to Theorem 3.3.2. Indeed, for any $a \in S$ there is a small neighbourhood U_a such that $U_a \cap S \Subset S$ where we still have flatness condition Eq. (3.6). Note that the topology used in Lemma 3.1.13 when restricted to the set of closed homogeneous subgroups is equivalent to the natural topology on the corresponding Grassmannian. Because S is connected and for some point W_p, $p \in S$, is a vertical subgroup of codimension N, the continuity of the map $a \to W_a$ forces W_a to be a vertical (so, in particular, normal) subgroup for all $a \in S$.

Thus, we deal with a continuous family $\{W_a \mid a \in S\}$ of vertical subgroups of codimension N for which, therefore, we can find a continuous map $k : S \to \mathrm{Hom}(\mathbb{G}, \mathbb{R}^N)$ such that $\operatorname{Ker} k(a) = W_a$ for every $a \in S$.

Fix an arbitrary compact subset $S' \Subset S$. Let us prove that $F \colon S' \to \mathbb{R}^N$, $F \equiv 0$, and k satisfy the condition of Whitney's extension Theorem 2.3.6. By contradiction, we can assume the existence of $\eta > 0$ and sequences a_m and b_m belonging to S' such that

- $a_m, b_m \to a \in S$ when $m \to \infty$,
- $|F(b_m) - F(a_m) - k(a_m)\langle a_m^{-1} \cdot b_m \rangle| = |k(a_m)\langle a_m^{-1} \cdot b_m \rangle| \geq \eta \mathsf{d}(a_m, b_m)$.

By the Reifenberg flatness of S', we can find $\tilde{b}_m \in a_m \cdot W_{a_m}$ such that $\mathsf{d}(\tilde{b}_m, b_m) \leq \epsilon(\mathsf{d}(a_m, b_m))\mathsf{d}(a_m, b_m)$. Since $\operatorname{Ker} k(a_m) = W_{a_m}$, we get a contradiction by

$$\begin{aligned} \eta \mathsf{d}(a_m, b_m) &\leq |k(a_m)\langle a_m^{-1} \cdot b_m \rangle| = |k(a_m)\langle a_m^{-1} \cdot \tilde{b}_m \cdot \tilde{b}_m^{-1} \cdot b_m \rangle| \\ &= |k(a_m)\langle \tilde{b}_m^{-1} \cdot b_m \rangle| \leq \|k(a_m)\| \epsilon(\mathsf{d}(a_m, b_m))\mathsf{d}(a_m, b_m), \end{aligned}$$

because $\epsilon(r) \to 0$, $r \to 0+$. Observe that the last argument uses only one-sided proximity in Reifenberg flatness: proximity of S' to its tangents and not the inverse one.

We see now that for any $S' \Subset S$ by extension we can associate $F_{S'} \in C_h^1(\mathbb{G}, \mathbb{R}^N)$ such that $S' \subset F^{-1}(0)$ and $D_h F_{S'} = k$ on S'. We must recall that Whitney's extension Theorem has the following locality property. There is a geometric constant $\infty > K > 0$ such that if $\mathsf{d}(x, S') = r$ then the value of the extension $F_{S'}(x)$ depends only on the initial data on $F \cap B(x, Kr)$. This can be seen in [VP06] by looking at the form of the so-called "extension" operator (Eq. 3.12, p. 610) and the properties of Whitney's type decomposition (Lemma 3, p. 608).

So, let $\{S_m \Subset S\}_{m \geq 0}$ be an increasing sequence of compact sets such that $\cup_m S_m = S$. We can arrange this sequence in such a way that S_m and $S_{m+2} \setminus S_{m+1}$ are η_m-separated with $\eta_m > 0$. Let $F_m \in C_h^1(\mathbb{G}, \mathbb{R}^m)$ be a map associated with S_m as above, i. e. $S_m \subset F^{-1}(0)$ and $D_h F_m = k$ on S_m. We define an open neighbourhood U_m of S_m as follows

$$U_m := \{a \in \mathbb{G} \mid \mathsf{d}(a, S_m) < (3K)^{-1}\eta_m\}, \quad m \geq 0.$$

By the locality properties of Whitney's extension and the triangle inequality, $F_n \equiv F_{m+1}$ on U_m for all $n \geq m + 1$. So, it is obvious that the sequence F_m converges to

$F \in C_h^1(U, \mathbb{R}^N)$ locally in $C_h^1(\tilde{U}, \mathbb{R}^N)$ where $\tilde{U} = \cup_m U_m$. It is also clear that $\tilde{U}$ is a neighbourhood of S and that $S \subset F^{-1}(0)$ with $D_h F(a) = k(a)$ for $a \in S$.

We use Theorems 3.1.1 and 3.3.2 to say that

$$\mathrm{pTan}^+_{\mathbb{G}}(F^{-1}(0), a) = \mathrm{pTan}^-_{\mathbb{G}}(F^{-1}(0), a) = \mathrm{Ker}\, D_h F(a) = \mathrm{Ker}\, k(a) = W_a$$

for any $a \in F^{-1}(0) \subset \tilde{U}$. Therefore, by Lemma 3.2.12 for any $a \in S$ there is a neighbourhood $U_a \subset \tilde{U}$ of a such that $S \cap U_a = F^{-1}(0) \cap U_a$. To conclude it is enough to take $U = \cup_{a \in S} U_a$. □

3.3.1. Characterisation of intrinsic sub-manifolds

Now we are going to derive some consequences of this local equivalence between uniform Reifenberg flatness and four cones coincidence. First we give a analogue of Theorem 3.1.1 in terms of paratangent cones.

Theorem 3.3.3. *Let $\mathbb{G}^1$, $\mathbb{G}^2$ be Carnot groups and let Ω be open in $\mathbb{G}^1$. Assume that $D_h F(p)$ is surjective at a point $p \in \Omega$ for some map $F \in C_h^1(\Omega, \mathbb{G}^2)$. Then there is a neighbourhood U of p in which the level set $S := F^{-1}(F(p))$ satisfies*

$$\mathrm{pTan}^+_{\mathbb{G}}(S, a) = \mathrm{pTan}^-_{\mathbb{G}}(S, a) = \mathrm{Ker}\, D_h F(a), \quad \forall\, a \in U \cap S.$$

In particular, if $\mathrm{Ker}\, D_h F(a)$ is not trivial, then has no isolated point.

Combining Theorems 3.1.1, 3.1.12 and 3.3.2 we can derive a new characterization of co-abelian intrinsic sub-manifolds in terms of their tangents.

Definition 3.3.4. A set $S \subset \mathbb{G}$ is called an co-abelian intrinsic sub-manifold of codimension N if in a neighbourhood of every $a \in S$, S coincides with a level set of a horizontally differentiable map $F \in C_h^1(\mathbb{G}, \mathbb{R}^N)$ with $D_h F(a)$ surjective.

Theorem 3.3.5 (Four cones theorem)**.** *Let $S \subset \mathbb{G}$ be a closed connected set. The following conditions are equivalent:*

1. *S is a co-abelian intrinsic sub-manifold of codimension N;*
2. *All four tangent cones coincide at every point $a \in S$:*

   $$\mathrm{pTan}^+_{\mathbb{G}}(S, a) = \mathrm{pTan}^-_{\mathbb{G}}(S, a),$$

   and there is some point $p \in S$ such that $\mathrm{pTan}^+_{\mathbb{G}}(S, p)$ is a vertical subgroup of codimension N.
3. *There is a family $\{W_a \mid a \in S\}$ of closed homogeneous sets such that W_p is a vertical subgroup of codimension N for some $p \in S$ and for every $S' \Subset S$ there is an increasing function $\epsilon\colon (0, \infty) \to (0, \infty)$, $\epsilon(t) \to 0+$ when $t \to 0+$, such that*

   $$\mathrm{dist}_{\mathsf{d}}\big(B(a, r) \cap S, B(a, r) \cap (a \cdot W_a)\big) \le \epsilon(r) r, \quad r > 0, \quad \forall\, a \in S'.$$

3.4. Some applications

3.4.1. Connectedness of level sets.

In a general situation, we are not able to describe the local topology of level sets. Meanwhile, we would like to conjecture the following

Conjecture 3.4.1. *Let $F \in C^1_h(\Omega, \mathbb{G}^2)$ be a map defined on open set $\Omega \subset \mathbb{G}^1$. Take a point $p \in \Omega$ such that $D_h F(p)$ is surjective. Then there is a neighbourhood $U \subset \Omega$ of p such that $F^{-1}(F(p)) \cap U$ is homeomorphic to* $\operatorname{Ker} D_h F(p)$.

The similar result was proved for Reifenberg flat sets in $\mathbb{R}^n$ ([Rei60]) using the Reifenberg parametrisation algorithm. This method fails to be directly applied in Carnot groups. One of the obvious reasons is that in general projections on vertical subgroup are not Lipschitz. Maybe, there are some other, more deep, reasons. Even if we strongly believe in Conjecture 3.4.1, that is not clear what kind of metric properties we should expect from a such homeomorphism. In particular, we don't think that for Reifenberg vanishing flat sets in Carnot groups it is always possible to obtain an almost bi-Lipschitz regularity in the contrast to Euclidean framework (see [DT99]). (For instance, one should try to check out the example of [Vit08, Th. 4.35] for a non-existence of β-Hölder homeomorphism with $\beta < 1$.)

Below we present only simple topological remarks.

Notation 3.4.2. In this section, for simplicity of calculations we are going to use a homogeneous norm of the form

$$\rho(\exp(\sum_i Y_i)) = \max_i k_i \|Y_i\|^{1/i}, \quad Y_i \in \mathfrak{g}_i, \tag{3.15}$$

with some positive parameters $\{k_i > 0\}$. It can be shown [FSS03a, Th. 5.1] that one can choose a set of these parameters in such a way that the left-invariant distance d built with ρ is a metric.

Proposition 3.4.3. *If $c = \exp(v) \in \mathbb{G}$ and $c' = \exp(v/2)$ (euclidean middle point between c and e) then,*

$$\mathsf{d}(c, c') = \mathsf{d}(e, c') \leq 2^{-\frac{1}{\deg v}} \mathsf{d}(c, e). \tag{3.16}$$

Proof. Vectors v and $v/2$ commute, so by Baker-Campbell-Hausdorff formula,

$$\log(c'^{-1} \cdot c) = (1 - \frac{1}{2})v = \log(e \cdot c'^{-1}).$$

Hence, Eq. (3.16) holds for any homogeneous norm of the form Eq. (3.15). □

Lemma 3.4.4. *Let $F \in C^1_h(\Omega; \mathbb{G}^2)$, $\Omega \subset \mathbb{G}^1$ open, be such that $D_h F(a_0)$ is surjective for some $a_0 \in \Omega$. Then there is a neighbourhood U of a_0 and $r_0 > 0$ such that any two points $a, b \in S \cap U$ with $\mathsf{d}_1(a, b) \leq r_0$ can be joined by a Hölder continuous curve Γ lying on $S := F^{-1}(F(a_0))$ whose diameter is uniformly controlled by $\mathsf{d}_1(a, b)^\beta$ with some $\beta > 0$. In particular, S is uniformly locally connected inside U.*

Proof. By Theorem 3.1.1, there is a neighbourhood U of a_0 where $S := F^{-1}(F(a_0))$ is Reifenberg vanishing flat w. r. t. the family of normal subgroups $\{W_p := \operatorname{Ker} D_h(p) \mid p \in S\}$. Let us take two points $a, b \in S \cap U$ with $\mathsf{d}_1(a, b) = r$ such that the modulus $\epsilon(r)$ in Theorem 3.1.1 is small. We are going to construct by dyadic iterations a continuous curve $\Gamma\colon [0, 1] \to S$ linking $\Gamma(0) = a$ and $\Gamma(1) = b$.

Let us explain one step of this dyadic construction. Take for $\tilde{b} \in a \cdot W_a$ a closest point to b which, therefore, satisfies $\mathsf{d}_1(b, \tilde{b}) < \epsilon(r)r$. By applying Proposition 3.4.3 to $c = a^{-1} \cdot \tilde{b} \in W_a$ we obtain a midpoint c' that still belongs to the subgroup W_a. and, if we put $\tilde{b}' = a \cdot c'$ then

$$\mathsf{d}_1(a, \tilde{b}') = \mathsf{d}_1(\tilde{b}', \tilde{b}) \leq 2^{-\frac{1}{\alpha}} \mathsf{d}_1(a, \tilde{b}),$$

where $\alpha = \max\{\deg X \mid X \in W_a\}$ does not depend on a. We define $\Gamma(1/2) = b'$ as a point of S closest to $\tilde{b}'$, which by flatness hypothesis satisfies $\mathsf{d}(b', \tilde{b}') \leq \epsilon(r)r$. By the triangle inequality,

$$\max\{\mathsf{d}(\Gamma(0), \Gamma(1/2)), \mathsf{d}(\Gamma(1/2), \Gamma(1))\} \leq \frac{\mathsf{d}(\Gamma(0), \Gamma(1))}{2^{1/\alpha} - 2\epsilon(r)}.$$

So, from the beginning we should take $r > 0$ in such a way that $2^{1/\alpha} - 2\epsilon(r) \geq K > 1$ with some fixed K. By the same procedure, we define $\Gamma(1/4)$ starting from $\Gamma(0)$ and $\Gamma(1/2)$, then $\Gamma(3/4)$ starting from $\Gamma(1/2)$ and $\Gamma(1)$, and so on for all dyadic points of $[0, 1]$. Note that in this construction, for $m \geq 1$ and $l = 0, \ldots, 2^m - 1$,

$$\mathsf{d}(\Gamma(l/2^m), \Gamma((l + 1)/2^m)) \leq \mathsf{d}(\Gamma(0), \Gamma(1))K^{-m}. \tag{3.17}$$

Since $K > 1$, a classical argument from [LV07, Lemma 2.] applied to Eq. (3.17), guarantees that the map Γ is Hölder continuous with exponent $\beta = \log K/ \log 2$ on dyadics. Therefore, Γ admits a continuous extension on $[0, 1]$ with values in S since S is closed. In fact, because of $\epsilon \to 0$, the curve $\Gamma\colon [0, 1] \to S$ will be Hölder continuous with any exponent β strictly less than α. □

The following result generalizes Theorem 5.3.7.

Theorem 3.4.5 (One-dimensional level sets). *Let $F \in C^1_h(\Omega, \mathbb{G}^2)$, Ω is open in $\mathbb{G}^1$, and $D_hF(a_0)$ is surjective and* $\operatorname{Ker} D_hF(a_0)$ *is one-dimensional (viewed as a linear space). Then there is a neighbourhood U of a_0 in which the level set $\Gamma := F^{-1}(F(a_0)) \cap U$ is a simple curve and $(\Gamma, \mathsf{d}_1^{\deg \operatorname{Ker} D_hF(a_0)})$ is a flat curve (as in Equation (5.15)).*

Remark 3.4.6. Note that assuming that Z is a normal homogeneous subgroup of linear dimension one implies that Z is a subset of the centre of $\mathbb{G}$. Indeed, if $a = \exp(X) \in Z$ and $b = \exp(Y) \in \mathbb{G}$, then by the classical formula for the adjoint representation of the Lie algebra, the element

$$b \cdot a \cdot b^{-1} = \exp(X + [Y, X] + \frac{1}{2!}[Y, [Y, X]] + \ldots)$$

must belong to Z due to its normality. But all commutator terms have homogeneous degree strictly bigger than X unless they are zero. So, $b \cdot a \cdot b^{-1} \in Z$ if and only if

$[Y, X] = 0$ for all $Y \in \mathfrak{g}$, which implies $Z \in \text{centre}(\mathbb{G})$. Thus, Theorem 3.4.5 can be applied, for instance, in the situation where $\mathbb{G}^1 = \mathbb{G}$ is a Carnot group and $\mathbb{G}^2 = \mathbb{G}/Z$ is the quotient of $\mathbb{G}$ by a one-dimensional homogeneous subgroup Z from the centre of $\mathbb{G}$.

Proof of Theorem 3.4.5. Thanks to Lemma 3.4.4, we can find an open ball $B(a_0, R)$, $R > 0$, such that any point of $S := F^{-1}(F(a_0)) \cap B(a_0, R)$ can be joined by a curve lying on $F^{-1}(F(a_0))$. Let us consider $T := \exp(\mathfrak{t})$ a homogeneous subspace that is complementary to $\operatorname{Ker} D_h F(a_0)$ as in the proof of Theorem 3.1.1. Note that due to the flatness of S, we can take a small radius $R > 0$ in such a way that

$$(a \cdot T) \cap S = \{a\} \text{ for any } a \in S. \tag{3.18}$$

Since $Z_a := \operatorname{Ker} D_h F(a)$ has (linear) dimension one, T is of co-dimension one, so that $\mathbb{G}^1 \setminus T$ has exactly two connected components that we will denote by T^+ and T^- (we put also $T_a^{\pm} = a \cdot T^{\pm}$). Thus, for any $a \in S$ we can decompose S as the following disjoint union

$$S = (S \cap T_a^+) \bigsqcup \{a\} \bigsqcup (S \cap T_a^-). \tag{3.19}$$

Now let us take two point $a, b \in S$. Let $\Gamma\colon [0, 1] \to S$ be an injective curve linking them. We are going to show that there is a neighbourhood of $\Gamma((0, 1))$ in which $\Gamma((0, 1))$ and S coincide. The argument is elementary: it consists of moving T along Γ and applying Eq. (3.18).

Indeed, take $c = \Gamma(t)$, $t \in (0, 1)$ any interior point. By Eq. (3.19) and the continuity of Γ, we can assume by changing the sign that $\Gamma([0, t)) \subset T_c^+$. For $s \in [0, 1]$ we introduce the set $T(s) = \Gamma(s) \cdot T$. By the choice of T, the intersection of $T(s)$ and $(c \cdot Z_c)$ consists of a unique point. This defines a continuous map $s \in [0, t] \to (c \cdot Z_c) \sim \mathbb{R}$. The image of this map covers a non-trivial closed interval in $(c \cdot Z_c) \cap T_c^+$ containing c. The interval is non-trivial because the intersection $T(s) \cap T(t)$ lies outside of some neighbourhood of c as soon as for $|s - t| \geq \delta$ with any fixed $\delta > 0$ (otherwise, it would contradict Eq. (3.18)). We can say the same about such a map $s \in [0, t] \to (d \cdot Z_c)$ for points $d \in T(t)$ close enough to c: the corresponding interval contains d and the length of those intervals can be bounded from below. Thus, there is a neighbourhood U of c such that $U \cap T_c^+ \subset \{T(s) \mid s \in [0, t]\}$. This means that by Eq. (3.18) in $U \cap T_c^+$ there is no point of S except $\Gamma([0, t))$. This also forces $\Gamma((t, 1))$ to lie in T_c^- where we can apply the same argument. Therefore, we are able to find a neighbourhood U of c such that $U \cap S \subset \Gamma((0, 1))$, and, thus, we are done.

Let us show that the curve Γ endowed with quasi-metric d^{α} is flat, $\alpha = \deg Z_a$. Let $a, b, c \in \Gamma$ such that $a \leq b \leq c$ w. r. t. a linear order on Γ. We use the Reifenberg flatness condition at the point b to find points $\tilde{a}, \tilde{c} \in b \cdot Z_b$ such that

$$\max\{\mathsf{d}(\tilde{a}, a), \mathsf{d}(\tilde{c}, c)\} \leq \epsilon(r) r,$$

where $r = \max\{\mathsf{d}(a, b), \mathsf{d}(b, c)\}$. Note that we can chose $\tilde{a}, \tilde{b}$ in such a way that $\tilde{a} \leq b \leq \tilde{c}$ w. r. t. an order on $b \cdot Z_b$. Indeed, since Z_b belongs to the centre of $\mathbb{G}$, the closest point map on it consists merely to take an orthogonal projection (in exponential coordinates). Next, by taking a power in the triangle inequality, we get that

$$\max\left\{|\mathsf{d}(\tilde{a},b)^\alpha - \mathsf{d}(a,b)^\alpha|, |\mathsf{d}(\tilde{c},b)^\alpha - \mathsf{d}(c,b)^\alpha|, |\mathsf{d}(\tilde{a},\tilde{c})^\alpha - \mathsf{d}(a,c)^\alpha|\right\} \lesssim \epsilon(r)r^\alpha,$$

so that,

$$|\mathsf{d}(a,b)^\alpha + \mathsf{d}(c,b)^\alpha - \mathsf{d}(a,c)^\alpha| \lesssim \epsilon(r)r^\alpha + |\mathsf{d}(\tilde{a},b)^\alpha + \mathsf{d}(b,\tilde{c})^\alpha - \mathsf{d}(\tilde{a},\tilde{c})^\alpha|.$$

As $\tilde{a} \leq b \leq \tilde{c}$, the last term is zero by a basic property of Euclidean distance $\mathsf{d}^\alpha \llcorner (b \cdot Z_b)$, this finishes the proof. □

3.4.2. Application to optimization problems

We present an application of the four cones theorem to optimality problems. In particular, we will consider a scalar function $G : \mathbb{G} \to \mathbb{R}$ and prove an intrinsic form of Peano's Regula for differentiable functions in the spirit of [DG07]. As a corollary, we obtain an intrinsic version of the Lagrange Multipliers theorem.

Theorem 3.4.7 (Peano's Regula)**.** *Let $G\colon \mathbb{G} \to \mathbb{R}$ be such that G is horizontally differentiable at $a_0 \in S$, where $S \subset \mathbb{G}$. If $G(a_0) = \max\{G(a) \mid a \in S\}$ (resp. $G(a_0) = \min\{G(a) \mid a \in S\}$), then*

$$D_hG(a_0)\langle v\rangle \leq 0, \qquad (\textit{resp. } D_hG(a_0)\langle v\rangle \geq 0,) \qquad \forall\, v \in \mathrm{Tan}^+_{\mathbb{G}}(S, a_0). \tag{3.20}$$

Proof. Let $v \in \mathrm{Tan}^+_{\mathbb{G}}(S, a_0)$. By definition, there exist two sequences $\{a_m\}_m \subset S$ and $\{t_m\}_m \subset \mathbb{R}_+$ such that (3.7) holds. As G is differentiable, by Proposition 3.2.7 we have

$$\lim_m \frac{G(a_m) - G(a_0)}{t_m} = D_hG(a_0)\langle v\rangle.$$

Since $G(a_0) = \max_S G$ it follows that $G(a_m) - G(a_0) \leq 0$ and thus we obtain Eq. (3.20). □

Theorem 3.4.8 (Lagrange multipliers)**.** *Let $F \in C^1_h(\mathbb{G}^1, \mathbb{G}^2)$ be such that D_hF is surjective at point $a_0 \in S := F^{-1}(e_2)$. Assume also that $a_0 \in S$ is such that $G(a_0) = \max\{G(a) \mid a \in S\}$ (resp. $G(a_0) = \min\{G(a) \mid a \in S\}$) for a scalar function $G\colon \mathbb{G}^1 \to \mathbb{R}$. If G is horizontally differentiable at a_0 then*

$$\mathrm{Ker}\, D_hF(a_0) \subset \mathrm{Ker}\, D_hG(a_0)$$

Proof. Theorem 3.3.5 implies that $\mathrm{Tan}^+_{\mathbb{G}}(S, a_0) = \mathrm{Ker}\, D_hF(a_0)$. Therefore, by Theorem 3.4.7 it follows that

$$D_hG(a_0)\langle v\rangle \leq 0 \qquad \forall\, v \in \mathrm{Ker}\, D_hF(a_0).$$

Since $\mathrm{Ker}\, D_hF(a_0)$ is bilateral, this implies that

$$D_hG(a_0)\langle v\rangle = 0 \qquad \forall\, v \in \mathrm{Ker}\, D_hF(a_0).$$

□

3.5. Dimension estimate

The main result of this section says that the Hausdorff dimension of level sets coincide with the Hausdorff dimension of their tangents. However, it is not true that the corresponding Hausdorff measure is finite or positive.

Theorem 3.5.1. *Let $F \in C_h^1(\Omega; \mathbb{G}^2)$, $\Omega \subset \mathbb{G}^1$ open, be such that $D_h F(a)$ is surjective for some $a \in \Omega$. Then there is a neighbourhood U of a in which the Hausdorff dimension of level set $\dim F^{-1}(F(a)) \cap U$ is equal to $\dim \mathbb{G}^1 - \dim \mathbb{G}^2$.*

This theorem is a consequence (by Remarks 3.5.7 and 3.5.8) of the general Theorem 3.5.6 and Reifenberg vanishing-flatness of the level sets, i. e. Theorem 3.1.1.

Let us first give some new definitions.

Definition 3.5.2. We say that metric space S belongs to $\mathcal{A}^\alpha(K_-, K_+, R_0)$ with positive parameters $\alpha > 0$, $K_- > 0$, $K_+ > 0$, $R_0 > 0$ if the following property holds:

1. every ball of radius $R \in (0, R_0]$ contains at most $K_+(R/r)^\alpha$ disjoint balls of radius $r > 0$;

2. every ball of radius $R \in (0, R_0]$ cannot be covered by less than $K_-(R/r)^\alpha$ balls of radius $r > 0$.

Remark. Observe that if properties 1. and 2. hold for some $R_0 > 0$ then they hold for any $R_0 > 0$ with some modified K_- and K_+.

Remark 3.5.3. If the Hausdorff measure $\mathcal{H}^\alpha$ is α-Ahlfors regular on some metric space S then (by merely (sub)additivity of measures) $S \in \mathcal{A}^\alpha$ with some constants R_0, K_+, K_- that can be expressed in terms of the Ahlfors regularity constants.

Let us explain now, following the approach from [Gro81], the notion Gromov-Hausdorff distance between metric spaces. For a metric space (S, d) and point $a \in S$, pointed metric space means simply a triple (S, d, a). For metric spaces (M_1, d_1) and (M_2, d_2) a metric ρ on the disjoint union $M_1 \sqcup M_2$ is called *admissible* if ρ coincides with d_1 on M_1 and with d_2 on M_2.

Definition 3.5.4. The *Gromov-Hausdorff distance* d_{GH} between any two pointed metric spaces (M_1, d_1, a_1) and (M_2, d_2, a_2) is defined as the infimum of all $\epsilon > 0$ such that there exists an admissible metric ρ on $M_1 \sqcup M_2$ for which

- $\rho(a_1, a_2) < \epsilon$;
- $\rho(M_1, a) < \epsilon$ for all $a \in B(a_1, \epsilon^{-1}) \cap M_2$;
- $\rho(M_2, a) < \epsilon$ for all $a \in B(a_2, \epsilon^{-1}) \cap M_1$.

Definition 3.5.5. A pointed metric space M belongs to the Gromov-Hausdorff tangent $\mathrm{Tan}_{GH}(S, p)$ to S at point p if there is a sequence $r_j \to 0+$ such that the rescaled pointed metric spaces $(S, d/r_j, p)$ converges as $j \to \infty$ to M in distance d_{GH}.

Theorem 3.5.6. *Let (S,d) be a complete metric space containing a dense countable subset. Assume that there are an increasing function $\epsilon\colon \mathbb{R}_+ \to \mathbb{R}_+$, $\epsilon(t) \searrow 0$ as $t \searrow 0+$, a real number $\alpha > 0$ and strictly positive constants K_-, K_+ and R_0 such that for every $p \in S$ and $0 < r \leq R_0$ one can find a pointed metric space $Z_{p,r} \in \mathcal{A}^\alpha(K_-, K_+, R_0)$ satisfying*

$$d_{GH}\big((S, d/r, p), Z_{p,r}\big) < \epsilon(r). \tag{3.21}$$

Then the Hausdorff dimension of S equal $\dim S = \alpha$*.*

Before we start the proof, let us add some remarks.

Remark 3.5.7. It is easy to check that Reifenberg vanishing flatness of level sets implies their uniform Gromov-Hausdorff convergence to the tangents. Of course, we should take $Z_{p,r} := (\operatorname{Ker} D_h F(p), \mathsf{d}_1, e_1)$ independently of scale r. So, $\operatorname{Tan}_{GH}(S,p)$ will consist of a unique element $Z_{p,r}$ (see Definition 3.5.5).

Indeed, since $Z_{p,r}$ is homogeneous, the dilated ambient metric d_1/r from $\mathbb{G}^1$ will be admissible for $(S, \mathsf{d}_1/r, p)$ and $Z_{p,r}$, and, if

$$\operatorname{dist}(B(p,r) \cap S, B(p,r) \cap (p \cdot \operatorname{Ker} D_h F(p))) \leq \epsilon(r) r, \quad r > 0,$$

then, immediately from the definition of d_{GH}, we get that

$$d_{GH}((S, \mathsf{d}_1/r, p), Z_{p,r}) \leq \tilde{\epsilon}(r),$$

where $\tilde{\epsilon}$ satisfies

$$\tilde{\epsilon}(r) = \epsilon(r/\tilde{\epsilon}(r)). \tag{3.22}$$

Using an elementary analysis one can see that for increasing and bounded ϵ, Eq. (3.22) has an increasing and bounded solution $\tilde{\epsilon}$, and if $\epsilon(0) = 0$ then $\tilde{\epsilon}(0) = 0$. In particular, if $\epsilon(r) \simeq r^\beta$ then $\tilde{\epsilon}(r) \simeq r^{\tilde{\beta}}$ with $\tilde{\beta} = \beta/(1+\beta)$.

Remark 3.5.8. All tangent spaces $Z_{p,r} = (\operatorname{Ker} D_h F(p), \mathsf{d}_1, e_1)$ belong to $\mathcal{A}^\alpha$ with uniform constants for p such that $D_h F(p)$ is surjective. In fact, since $Z_{p,r}$ is a homogeneous subgroup in $\mathbb{G}$, the Hausdorff measure $\mathcal{H}^\alpha(B(a,R))$ of any ball is equal to $C_p R^\alpha$. This implies that we can take $K_- = K_+ = 1$ and $R_0 = \infty$.

Proof of Theorem 3.5.6. To estimate the Hausdorff measure we are going to use a multi-scale analysis technique. It is known (see, for instance, [HT13, Th. 2.4]) that for any $\delta \in (0, 1/2)$ there exists a filtration of maximal nets $\{\mathcal{A}_k \subset S\}_{k \in \mathbb{Z}}$ such that

$$\begin{gathered}
\mathcal{A}_k \subseteq \mathcal{A}_{k+1}, \\
d(a,b) \geq \delta^k \text{ for } a, b \in \mathcal{A}_k \text{ with } a \neq b, \\
d(S, \mathcal{A}_k) < \delta^k.
\end{gathered}$$

Let us fix $\delta \in (0, 1/2)$ and a such filtration $\{\mathcal{A}_k\}$.

Let us fix an arbitrary $r > 0$ and $p \in S$. By the definition of d_{GH} and the triangle inequality, there is a map $t : B(p, r/\epsilon(r)) \subset S \to Z_{p,r}$ (not necessarily continuous) such that for every pair of points $a, b \in B(p, r/\epsilon(r))$

$$|r^{-1} d(a,b) - d_p(t(a), t(b))| < 2\epsilon(r), \tag{3.23}$$

where d_p denotes the metric on $Z_{p,r}$. Observe that we can assume p being mapped by t to the marked point of $Z_{p,r}$ denoted by $\bar{p}$. Symmetrically, for every $r > 0$ there is a map $\bar{t} : B(\bar{p}, 1/\epsilon(r)) \subset Z_{p,r} \to S$ such that for every pair of points $a, b \in B(\bar{p}, 1/\epsilon(r))$

$$|r^{-1}d(\bar{t}(a), \bar{t}(b)) - d_p(a, b)| < 2\epsilon(r). \tag{3.24}$$

Obviously, we can also assume that $\bar{t}(t(a)) = a$ if $t(a) \in B(\bar{p}, 1/\epsilon(r))$.

Now, let us estimate the number of points of $\mathcal{A}_k$ inside a ball.

Proposition 3.5.9. *Put $\mathcal{A}_j(p, R) := B(p, R) \cap \mathcal{A}_j$. Then*

$$K_- \left(\frac{R}{\delta^j + 2\epsilon(R)R} - 1 \right)^\alpha \leq \#\mathcal{A}_j(p, R) \leq K_+ \left(\frac{R + \delta^j}{\delta^j - 2\epsilon(R)R} \right)^\alpha, \tag{3.25}$$

provided that $\delta^j > 2\epsilon(R)R$ and $\epsilon(R) < 1$.

Proof. Put $r/\epsilon(r) = R$, note that $R > r$. We deduce from Eq. (3.23) that

$$d_p(t(a), t(b)) > \delta^j/r - 2\epsilon(r), \quad a, b \in \mathcal{A}_j(p, R).$$

This means that the balls $\{B(t(a), \delta^j/r - 2\epsilon(r)) \mid a \in \mathcal{A}_j(p, R)\}$ are pair-wise disjoint. All these balls are contained in the ball $B(t(p), R')$ where

$$R' = \max_{a \in \mathcal{A}_j(p,R)} d_p(t(p), t(a)) + \frac{\delta^j}{r} - 2\epsilon(r) \leq \frac{R}{r} + 2\epsilon(r) + \frac{\delta^j}{r} - 2\epsilon(r) = \frac{R + \delta^j}{r}.$$

That is where the upper bound comes from.

In order to find the lower bound, let us show that the balls $\{B(t(a), \delta^j/r + 2\epsilon(r)) \mid a \in \mathcal{A}_j(p, R)\}$ cover $B(\bar{p}, \bar{R})$ where $\bar{R} = (R - \delta^j)/r - 2\epsilon(r)$. Note that $\bar{R}$ is chosen in such a way that $\bar{t}(B(\bar{p}, \bar{R})) \subset B(p, R - \delta^j)$ (see Eq. (3.24)). Observe also that the balls $\{B(a, \delta^j) \mid a \in \mathcal{A}_j(p, R)\}$ cover $B(p, R - \delta^j)$ because it cannot be reached from $S \setminus \mathcal{A}_j(p, R)$. So, if for some $b \in B(\bar{p}, \bar{R})$

$$d_p(b, t(a)) > \delta^j/r + 2\epsilon(r),$$

then by Eq. (3.24), $d(\bar{t}(b), a) > \delta^j$ that is not possible for all $a \in \mathcal{A}_j(p, R)$. $\square$

Let us take the starting scale j_0 such that $\epsilon(\delta^{j_0})$ is small enough, say, $\epsilon(\delta^{j_0-2}) < \delta^{-10}$. We can define an increasing sequence $\{j_i\}_{i\geq 0}$ in such a way that $2\delta^{j_i - j_{i+1}}\epsilon(\delta^{j_i-1}) \in (1/4, 1/2]$. Observe that since ϵ is increasing and tends to 0 at 0, the "scale jump" $\Delta_i j := j_{i+1} - j_i \geq 1$ is increasing and tends to ∞ with $i \to \infty$.

Let $j > k > i$ and $p \in S$. We know that $B(p, \delta^i)$ is covered by $\{B(a, \delta^k) \mid a \in \mathcal{A}_k(p, \delta^i + \delta^k)\}$, so,

$$\#\mathcal{A}_j(p, \delta^i) \leq \sum_{a \in \mathcal{A}_k(p, \delta^i + \delta^k)} \#\mathcal{A}_j(a, \delta^k) \leq \#\mathcal{A}_k(p, \delta^i + \delta^k) \max_{a \in \mathcal{A}_k(p, \delta^i + \delta^k)} \#\mathcal{A}_j(a, \delta^k).$$

We recursively apply this argument following the scales j_i and we use the estimates Eq. (3.25) (that do not depend on the centre of ball) to count relative number of points between two scales. We get that

$$\begin{aligned}
\#\mathcal{A}_{j_N}(p,\delta^{j_0}) &\le \prod_{i=0}^{N-1} K_+\left(\frac{\delta^{j_i}+2\delta^{j_{i+1}}}{\delta^{j_{i+1}}-2\epsilon(\delta^{j_i-1})\delta^{j_i-1}}\right)^{\alpha}\\
&\le \prod_{i=0}^{N-1} K_+\left(\frac{\delta^{-\Delta_i j}+2}{1-2\epsilon(\delta^{j_i-1})\delta^{-\Delta_i j-1}}\right)^{\alpha}\\
&\le \prod_{i=0}^{N-1} K_+ C\delta^{-\alpha\Delta_i j} = \delta^{-\alpha(j_N-j_0)}\prod_{i=0}^{N-1} K_+ C,
\end{aligned}$$

where $C\in[1,\infty)$ is an absolute constant because of the choice of the sequence j_i (note that $\delta^{j_i}+\delta^{j_{i+1}}\le\delta^{j_i-1}$ and additive constant 2 is negligible w. r. t. $\delta^{-\Delta_i j}$). For $j\ge j_0$, let $N(j)$ be an index such that $j\in(j_N,j_{N+1}]$. We use again Eq. (3.25) to count balls of scale j inside a ball of scale j_N, i. e.

$$\#\mathcal{A}_j(a,\delta^{j_N})\le K_+\left(\frac{\delta^{j_N}+\delta^{j}}{\delta^{j}-2\epsilon(\delta^{j_N})\delta^{j_N}}\right)^{\alpha}\le K_+\left(\frac{\delta^{j_N-j}+1}{1-2\epsilon(\delta^{j_N})\delta^{j_N-j}}\right)^{\alpha},\quad a\in\mathcal{A}_{j_N},$$

Since $j\le j_{N+1}$, the denominator is larger than

$$1-2\epsilon(\delta^{j_N})\delta^{j_N-j}\ge 1-2\epsilon(\delta^{j_N})\delta^{j_N-j_{N+1}}\ge\frac{3}{4},$$

so by passing from scale j_N to j we loose only a constant:

$$\#\mathcal{A}_j(a,\delta^{j_0})\le CK_+\delta^{-\alpha(j-j_0)}\prod_{i=0}^{N-1}K_+C.$$

If $\alpha'>\alpha$, then, since $\Delta_i j$ is increasing and tends to ∞,

$$\delta^{\alpha'(j-j_0)}\#\mathcal{A}_{j_N}(p,\delta^{j_0})\le CK_+\prod_{i=0}^{N(j)-1}K_+C\delta^{(\alpha'-\alpha)\Delta_i j}\to 0,\quad j\to\infty, \tag{3.26}$$

because for large i the factors in the last product are less than and bounded away from 1. Thus, if $\alpha'>\alpha$, the Hausdorff measure $\mathcal{H}^{\alpha'}$ of any small ball $B(p,\delta^{j_0})$ equals 0.

To show that $\mathcal{H}^{\alpha'}(B(p,\delta^{j_0}))=\infty$ when $\alpha'<\alpha$, we will define a probability measure μ on $B(p,\delta^{j_0})$ such that $\mu(B(a,r))r^{-\alpha'}\to 0$ when $r\to+\infty$ for all a. Note that this argument requires the completeness of S. We know that balls $\{B(a,\delta^k/2)\mid a\in\mathcal{A}_k\}$ are pair-wise disjoint, and so,

$$\#\mathcal{A}_j(p,\delta^i)\ge\sum_{a\in\mathcal{A}_k(p,\delta^i-\frac{\delta^k}{2})}\#\mathcal{A}_j(a,\frac{\delta^k}{2})\ge\#\mathcal{A}_k(p,\delta^i-\frac{\delta^k}{2})\min_{a\in\mathcal{A}_k(p,\delta^i-\frac{\delta^k}{2})}\#\mathcal{A}_j(a,\frac{\delta^k}{2}).$$

The measure μ will be constructed as a mass distribution. First, total mass 1 of $B(p, \delta^{j_0})$ is uniformly shared among balls $\{B(a, \delta^{j_1}/2) \mid a \in \mathcal{A}_{j_1}(p, \delta^{j_0})\}$, then the mass of each $B(a, \delta^{j_1}/2)$ is uniformly shared among balls $\{B(b, \delta^{j_2}/2) \mid b \in \mathcal{A}_{j_2}(a, \delta^{j_1}/2)\}$, and so on. By an argument similar to the above upper bound, we can get a lower bound for the number of balls of radius $\delta^{j_N}/2$ supporting μ, so that,

$$\mu(B(a, \delta^{j_N}/2))^{-1} \geq \delta^{-\alpha(j_N - j_0)} \prod_{i=0}^{N-1} K\_C^{-1}, \quad C > 1.$$

(Using balls of radius $\delta^{j_k}/2$ instead of δ^{j_k} influences only the constant C.) For $j \in (j_{N(j)}, j_{N(j)+1}]$, we apply the lower bound of Eq. (3.25), and as before, we loose a constant in our estimates

$$\mu(B(a, \delta^{j}/2))^{-1} \geq \delta^{-\alpha(j - j_0)} \prod_{i=0}^{N(j)} K\_C^{-1}.$$

Now, it is easy to see that μ has the required behaviour.

To conclude (i. e. to pass from small balls to the whole set) we also need the technical hypothesis that S has a countable dense subset. □

3.5.1. Other concepts of metric dimension

At the end of the section we are going to discuss some other concepts of dimension. Let (M, d) be a metric space, and let E be a non-empty subset of M. For $r > 0$, let $N_r(E)$ denote the least number of metric open balls of radius less than or equal to r with which it is possible to cover the set E.

Definition 3.5.10 ([Ass79])**.** The *Assouad dimension* of E is defined to be the infimum of $\alpha \geq 0$ for which there exist positive constants C and ρ such that, whenever $0 < r < R \leq \rho$, the following bound holds:

$$\sup_{x \in E} N_r(B(x, R) \cap E) \leq C \left(\frac{R}{r}\right)^{\alpha}.$$

Definition. The lower and upper *box-counting dimension* (or *Minkowski-Bouligand dimension*) of E are defined respectively as the lower and upper limits when $r \to 0+$ of

$$-\frac{\log(N_r(E))}{\log(r)}.$$

If those two dimensions coincide, we can speak merely about box-counting dimension.

The basic relation between these dimensions can be stated as follows.

Proposition ([Luu98, Th. A.5])**.** *For any metric space S,*

$$\dim_{\text{Hausdorff}} S \leq \dim_{\text{Assouad}} S.$$

and if S is bounded,

$$\dim_{\text{Hausdorff}} S \leq \underline{\dim}_{\text{box-counting}} S \leq \overline{\dim}_{\text{box-counting}} S \leq \dim_{\text{Assouad}} S.$$

Remark 3.5.11. As the Hausdorff dimension, the Assouad dimension of a closed homogeneous subgroup $\mathbb{W} \leq \mathbb{G}$ is equal to its homogeneous dimension $\alpha = \dim W$. More generally, if $M \in \mathcal{A}^\alpha$ then it follows from the definition that

$$\alpha = \dim_{\mathrm{Assouad}} M.$$

Moreover, $\dim_{\mathrm{Hausdorff}} M = \alpha$ if M is complete and contains a dense countable subset. The lower bound follows immediately from Theorem 3.5.6 with $\epsilon \equiv 0$. Indeed, we can consider as $Z_{r,p}$ the rescaled pointed space itself that belongs to $\mathcal{A}^\alpha$ with the same constants for any $r \leq 1$. Note also that the Hausdorff measure of any small ball (of radius less than R_0) in M is finite.

Proposition 3.5.12. *Under the hypothesis of Theorem 3.5.6, the Assouad dimension of S is equal to α, and if S is bounded, the box-counting dimension of S is equal to α.*

Proof. Using Eq. (3.26) it is easy to derive that the Assouad dimension is less than α' for any $\alpha' > \alpha$. The rest follows from basic relations between the dimensions. □

In particular, the Assouad dimension and Hausdorff dimension of level sets coincide and are equal to the homogeneous dimension of their tangents, i. e. $\dim \mathbb{G}^1 - \dim \mathbb{G}^2$.

In a recent preprint [DR13], the authors consider similar assumptions on metric spaces with respect to their tangents. Concerning the Assouad dimension they show the following result.

Theorem ([DR13, Th. 1.4])**.** *Let M be a metric space which admits at every point a single tangent space. Let $M' \subseteq M$ be a relatively compact set. Assume that the convergence towards the tangents is uniform on M'. Then*

$$\sup_{a \in \operatorname{int} M'} \dim_{\mathrm{Assouad}} T_x M \leq \dim_{\mathrm{Assouad}} M' \leq \sup_{a \in \operatorname{cl} M'} \dim_{\mathrm{Assouad}} T_x M.$$

Compared to this result, Theorem 3.5.6 requires stronger assumptions on tangent spaces (in particular, a uniform lower bound in Definition 3.5.2) and gives a stronger conclusion by providing a lower bound for the Hausdoff dimension.

In the same context, we should also consider the Nagata dimension. We continue following the presentation given in [DR13].

Definition (Nagata dimension)**.** Let (M, d) be a metric space. The *Nagata dimension*, or Assouad-Nagata dimension, of M is denoted by $\dim_N M$ and is defined as the infimum of all integers n with the following property: there exists a constant $c > 0$ such that, for all $s > 0$, M admits a cs-bounded cover with s-multiplicity at most $n + 1$.

We explain the terminology. Two subsets $A, B \subset M$ are *s-separated*, for some constant $s \geq 0$, if $\inf\{d(a, b) \mid a \in A, b \in B\} \geq s$. A family of subsets is called *s-separated* if each distinct pair of elements in it is s-separated. Let $\mathcal{B}$ be a cover of a metric space M. Then, for $s > 0$, the *s-multiplicity* of $\mathcal{B}$ is the infimum of all n such that every subset of M with diameter at most s meets at most n members of the family $\mathcal{B}$. Furthermore, $\mathcal{B}$ is called *D-bounded*, for some constant $D \geq 0$, if $\operatorname{diam} B \leq D$ for all $B \in \mathcal{B}$.

The Nagata dimension is invariant under quasi-symmetric homeomorphism.

Theorem ([LS05, Th. 1.2.]). *If $f\colon (X,\rho_X) \to (Y,\rho_Y)$ is a quasi-symmetric homeomorphism of metric spaces, then $\dim_N X = \dim_N Y$.*

The Nagata dimension is always greater or equal to the topological dimension. The basic relation between the Nagata and the Assouad dimensions is the following.

Theorem ([DR13, Th. 1.1]). *For all metric spaces M, the Nagata dimension of M is less than or equal to the Assouad dimension of M.*

Regarding the tangents, the following results holds.

Theorem ([DR13, Th. 1.2]). *Let M be a metric space which admits at every point a single tangent space. Let $M' \subseteq M$ be a relatively compact set with $\dim_N M' < \infty$. Assume that the convergence toward the tangents is uniform on the closure of M'. Then we have*

$$\sup_{x\in \operatorname{int} M'} \dim_N T_x M \le \dim_N M' \le \sup_{x\in \operatorname{cl} M'} \dim_N T_x M.$$

Lang and Le Donne (see [DR13, Th. 4.2]) proved that the Nagata dimension of a Carnot group $(\mathbb{G}, \mathsf{d})$ equals its topological dimension. Exactly the same argument can be applied to show that any closed homogeneous subgroup $W \subset \mathbb{G}$ endowed with a metric d induced from $\mathbb{G}$ has Nagata dimension equal to its topological dimension. This leads to the following result about level sets.

Corollary 3.5.13. *With the notation and assumptions of Theorem 3.5.1, the Nagata dimension of level sets is equal to the topological dimension of its tangents, that is*

$$\dim_N F^{-1}(F(a)) \cap U = \dim_N \mathbb{G}^1 - \dim_N \mathbb{G}^2.$$

It is also worth mentioning some general relation between the Nagata dimension of level sets of Lipschitz maps and the Nagata dimensions of target and source spaces.

Theorem ([DH08, Th. 3.4]). *Suppose $f\colon (X,\rho_X) \to (Y,\rho_Y)$ satisfies $\rho_X(x, f^{-1}(y)) \le \rho_Y(f(x), y)$ for all $x \in X$ and $y \in Y$. If f is Lipschitz and*

$$\dim_N(f^{-1}(y)) \le k$$

uniformly [2] with respect to $y \in f(X)$, then $\dim_N(X) \le k + \dim_N(Y)$.

As an application of the last result, it can be shown (see [DH08, Pr. 5.1]) that the Nagata dimension of the discrete Heisenberg group $\mathbb{H}^1(\mathbb{Z})$ equals 3.

[2]The concept of $\dim_N(A_s) \le n$ *uniformly with respect to* $s \in S$ means that the constant c in the definition of Nagata dimension can be chosen the same for all A_s, $s \in S$.

4. On graphs in Carnot groups

This chapter is devoted to the study of splittings of Carnot groups into semidirect products and of the natural objects appearing in this context. As a main result, we present a new universal characterization, Theorem 4.2.16, of Lipschitz graphs in terms of the trajectory-wise metric behaviour of a graph-map. We use Theorem 4.2.16 to derive a new characterization, Theorem 4.3.1, of co-abelian surfaces that can be represented as a graph. We also underline the difference between those characterizations and corresponding characterizations in two-step Carnot groups.

4.1. Splitting of Carnot groups

Definition 4.1.1. Let $\mathbb{K} \trianglelefteq \mathbb{G}$ be a normal homogeneous subgroup. A homogeneous subgroup $\mathbb{H} < \mathbb{G}$ is said to be *complementary* to $\mathbb{K}$ if $\mathbb{G}$ is a semidirect product of $\mathbb{K}$ and $\mathbb{H}$, i.e. $\mathbb{K} \cap \mathbb{H} = e$ and any element $a \in \mathbb{G}$ can be written as $a = n \cdot h$ with $n \in \mathbb{K}$ and $h \in \mathbb{H}$.

Notation 4.1.2. Let $\mathbb{G}$ be split into a semidirect product $\mathbb{G} = \mathbb{K} \ltimes \mathbb{H}$ of two homogeneous subgroups, where $\mathbb{K}$ is normal. We introduce the projections on the splitting factors:

$$\pi_{\mathbb{K}} \colon \mathbb{G} \to \mathbb{K}, \quad \pi_{\mathbb{H}} \colon \mathbb{G} \to \mathbb{H},$$

so that $\mathbb{G} \ni a = \pi_{\mathbb{K}}(a) \cdot \pi_{\mathbb{H}}(a)$. We denote by $\mathfrak{k} = \log(\mathbb{K})$ (it is a homogeneous ideal) and by $\mathfrak{h} = \log(\mathbb{H})$ (it is homogeneous sub-algebra of $\mathfrak{g}$). It is clear that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}$ and we denote the projections on the respective subspaces by

$$\pi_{\mathfrak{k}} \colon \mathfrak{g} \to \mathfrak{k}, \quad \pi_{\mathfrak{h}} \colon \mathfrak{g} \to \mathfrak{h}.$$

Of course, any tangent space can be decomposed accordingly, i. e. $T_a\mathbb{G} = T_a\mathbb{K} \oplus T_a\mathbb{H}$, $a \in \mathbb{G}$, where $T_a\mathbb{K} = dL_a(e)\langle\mathfrak{k}\rangle$ and $T_a\mathbb{H} = dL_a(e)\langle\mathfrak{h}\rangle$ are left-invariant distributions.

Let us recall basic facts about semidirect products. First we list abstract algebraic properties:

- for every $a \in \mathbb{G}$, a representation $a = n \cdot h$ with $n \in \mathbb{K}, h \in \mathbb{H}$ is unique;
- every element $a \in \mathbb{G}$ admits also a unique representation as a product $a = h \cdot n$ with $n \in \mathbb{K}, h \in \mathbb{H}$;
- $\pi_{\mathbb{H}}$ is a group homomorphism that is identity on $\mathbb{H}$ and its kernel is $\mathbb{K}$;
- the subgroup $\mathbb{H}$ is a naturally isomorphic to the quotient $\mathbb{G}/\mathbb{K}$.

In fact, each of these properties can be taken as an equivalent definition of semidirect product.

Proposition 4.1.3. *Let $\mathbb{H}$ be complementary to $\mathbb{K}$ in a Carnot group $\mathbb{G}$. Then the properties below hold.*

1. *Both maps $\pi_{\mathbb{H}}$ and $\pi_{\mathbb{K}}$ are idempotent and homogeneous and*
$$\pi_{\mathfrak{k}} = d\pi_{\mathbb{K}}(e), \quad \pi_{\mathfrak{h}} = d\pi_{\mathbb{H}}(e).$$

2. *Projection $\pi_{\mathbb{H}}$ is a group homomorphism and projection $\pi_{\mathbb{K}}$ satisfies*
$$\pi_{\mathbb{K}}(a \cdot b) = \pi_{\mathbb{K}}(a) \cdot \operatorname{Conj}_{\pi_{\mathbb{H}}(a)}(\pi_{\mathbb{K}}(b)),$$
where $\operatorname{Conj}_g(h) := g \cdot h \cdot g^{-1}$ is a group conjugation. In particular,
$$\begin{aligned}\pi_{\mathbb{H}}(a^{-1}) &= \pi_{\mathbb{H}}(a)^{-1},\\ \pi_{\mathbb{K}}(a^{-1}) &= \operatorname{Conj}_{\pi_{\mathbb{H}}(a)^{-1}}(\pi_{\mathbb{K}}(a)^{-1}).\end{aligned}$$

3. $\exp \circ \pi_{\mathfrak{h}} = \pi_{\mathbb{H}} \circ \exp$.

4. *The homogeneous subgroup $\mathbb{H}$ is itself a Carnot group.*

Proof. The first property is rather obvious, let us check the others. Take $a, b \in \mathbb{G}$. Then, by definition,
$$\pi_{\mathbb{K}}(a) \cdot \pi_{\mathbb{H}}(a) \cdot \pi_{\mathbb{K}}(b) \cdot \pi_{\mathbb{H}}(b) = a \cdot b = \pi_{\mathbb{K}}(a \cdot b) \cdot \pi_{\mathbb{H}}(a \cdot b).$$
Therefore, by uniqueness, the following must hold:
$$\begin{aligned}\pi_{\mathbb{K}}(a \cdot b) &= \pi_{\mathbb{K}}(a) \cdot \pi_{\mathbb{H}}(a) \cdot \pi_{\mathbb{K}}(b) \cdot \pi_{\mathbb{H}}(a)^{-1},\\ \pi_{\mathbb{H}}(a \cdot b) &= \pi_{\mathbb{H}}(a) \cdot \pi_{\mathbb{H}}(b).\end{aligned}$$
Note that because $\mathbb{K}$ is normal, $\operatorname{Conj}_{\pi_{\mathbb{H}}(a)}(\pi_{\mathbb{K}}(b)) = \pi_{\mathbb{H}}(a) \cdot \pi_{\mathbb{K}}(b) \cdot \pi_{\mathbb{H}}(a)^{-1}$ still belongs to $\mathbb{K}$. This proves 2.

Since $\pi_{\mathbb{H}}$ is a group homomorphisms, 3. is merely a basic property of the exponential map. (Note that we also deduce this directly using the fact that $\mathfrak{k}$ is an ideal ($[\mathfrak{k}, \mathfrak{g}] \subset \mathfrak{k}$) in Baker-Campbell-Hausdorff formula.)

Since $\pi_{\mathfrak{h}}$ is a homogeneous homomorphism of Lie algebras and $\mathfrak{g}_1$ (Lie bracket) generates $\mathfrak{g}$, then $\mathfrak{h}_1 = \pi_{\mathfrak{h}}(\mathfrak{g}_1)$ generates $\mathfrak{h} = \pi_{\mathfrak{h}}(\mathfrak{g})$, i. e. $\mathbb{H}$ is a Carnot group. □

Notation 4.1.4. We introduce the map $\sigma_a \colon \mathbb{K} \to \mathbb{K}$, $a \in \mathbb{G}$, defined by
$$\sigma_a = L_{\pi_{\mathbb{K}}(a)} \circ \operatorname{Conj}_{\pi_{\mathbb{H}}(a)}.$$

Remark 4.1.5. The map σ_a is a composition of right and left translations, therefore for any $a \in \mathbb{G}$ it preserves a volume (Haar) measure on $\mathbb{K}$.

Proposition 4.1.3 says in particular that the following diagrams commute:

$$
\begin{tikzcd}
\mathbb{H} \arrow[r, "\delta_r"] & \mathbb{H} \\
\mathbb{G} \arrow[u, "\pi_{\mathbb{H}}"] \arrow[r, "\delta_r"] \arrow[d, "\pi_{\mathbb{K}}"'] & \mathbb{G} \arrow[u, "\pi_{\mathbb{H}}"'] \arrow[d, "\pi_{\mathbb{K}}"'] \\
\mathbb{K} \arrow[r, "\delta_r"'] & \mathbb{K}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{H} \arrow[r, "L_{\pi_{\mathbb{H}}(a)}"] & \mathbb{H} \\
\mathbb{G} \arrow[u, "\pi_{\mathbb{H}}"] \arrow[r, "L_a"] \arrow[d, "\pi_{\mathbb{K}}"'] & \mathbb{G} \arrow[u, "\pi_{\mathbb{H}}"'] \arrow[d, "\pi_{\mathbb{K}}"'] \\
\mathbb{K} \arrow[r, "\sigma_a"'] & \mathbb{K}
\end{tikzcd}
. \tag{4.1}
$$

Remark 4.1.6. Because the inverse of L_a is $L_{a^{-1}}$, the map σ_a is invertible with inverse given by $\sigma_{a^{-1}}$:

$$\sigma_{a^{-1}}(b) = L_{\pi_{\mathbb{K}}(a^{-1})} \circ \mathrm{Conj}_{\pi_{\mathbb{H}}(a^{-1})}(b) = a^{-1} \cdot b \cdot \pi_{\mathbb{H}}(a).$$

4.1.1. Some examples.

Example 4.1.7 ([Mag13, Corollary 11.24])**.** The direct product of two Heisenberg groups has a unique non-trivial factorization.

Example 4.1.8 ([FSS07, Lemma 3.26])**.** A vertical subgroup $\mathbb{K}$ in Heisenberg group $\mathbb{H}^n$ admits a complementary (horizontal) subgroup if and only if $\dim \mathfrak{k} \cap \mathfrak{g}_1 \leq n$.

Example 4.1.9. Any vertical subgroup of codimension one has a one-dimensional complementary subgroup.

Observe, that except from one codimensional case, the property "to have a complementary subgroup" is not codimension invariant.

Example 4.1.10. Consider the stratified two-step Lie algebra of dimension 5

$$\mathfrak{g} = \mathrm{span}\{X, Y_1, Y_2\} \oplus \mathrm{span}\{Z_1, Z_2\},$$

with non-trival commutators given by $[X, Y_1] = Z_1$ and $[X, Y_2] = Z_2$. Inside $\mathfrak{g}$ there are two types of vertical sub-algebras $\mathfrak{k} \subset \mathfrak{g}$ of codimension 2:

- if $\mathfrak{k} \cap \mathrm{span}\{Y_1, Y_2\} \neq \{0\}$, then $\mathfrak{k}$ does not admit a complementary (horizontal) sub-algebra ($\mathrm{span}\{Y_1, Y_2\}$ is the only horizontal sub-algebra);
- otherwise, it does (the complementary sub-algebra is just $\mathrm{span}\{Y_1, Y_2\}$).

Thus, in the Carnot group $\exp(\mathfrak{g})$, the set of vertical subgroups of codimension 2 admitting a complementary subgroup is open and dense in the natural topology, but different from the whole set of vertical subgroups of codimension 2.

4.1.2. Compatible Coordinates.

Sometimes we will need to work in coordinates. For this purpose, we fix (as in Section 2.1.1) a basis of homogeneous left-invariant vector fields, denoted by $\{Y_i\}_{i=1}^N \subset \mathfrak{k}$ and $\{X_i\}_{i=1}^M \subset \mathfrak{h}$, $(N + M = \dim \mathfrak{g})$, such that

$$T\mathbb{K} = \operatorname{span}\{Y_i\}, \quad T\mathbb{H} = \operatorname{span}\{X_j\}.$$

We also assume that $\deg Y_i$ and $\deg X_i$ are increasing functions of i. Thus, we obtain an exponential coordinate system on $\mathbb{G}$ compatible with the splitting $\mathbb{G} = \mathbb{K} \ltimes \mathbb{H}$:

$$\mathbb{G} \ni a = \exp\left(\sum_{i=1}^N y_i Y_i + \sum_{j=1}^M x_j X_j\right), \quad a \equiv (\mathbf{y}, \mathbf{x}) := (y_1, \ldots, y_N, x_1, \ldots, x_M).$$

In particular, $e = (0, 0)$ and $a^{-1} = (-\mathbf{y}, -\mathbf{x})$. By abuse of notation, we will often write $\mathbf{x}$ instead of $(0, \mathbf{x})$ and $\mathbf{y}$ instead of $(\mathbf{y}, 0)$ to denote the elements in $\mathbb{G}$ lying in $\mathbb{H}$ and $\mathbb{K}$ respectively. (Note that in general $\mathbf{y} \cdot \mathbf{x} \neq (\mathbf{y}, \mathbf{x})$.) For example, the group operation given by Baker–Campbell–Hausdorff formula will be polynomial in these coordinates and can be represented in the following form:

$$(\mathbf{y}, \mathbf{x}) \cdot (\mathbf{y}', \mathbf{x}') = \big(Q_y(\mathbf{y}, \mathbf{y}', \mathbf{x}, \mathbf{x}'), Q_x(\mathbf{x}, \mathbf{x}')\big).$$

Here, the polynomial $Q_x = (Q_x^1, \ldots, Q_x^M)$ does not depend on $(\mathbf{y}, \mathbf{y}')$ due to property 3. in Proposition 4.1.3.

4.1.3. Base projector.

Definition 4.1.11. Let a splitting $\mathbb{G} = \mathbb{K} \ltimes \mathbb{H}$ be given. For every $a \in \mathbb{G}$ we define a linear map

$$\Pi_a := d\pi_{\mathbb{K}}(a), \quad \Pi_a \colon T_a\mathbb{G} \to T_{\pi_{\mathbb{K}}(a)}\mathbb{K},$$

that we shall call *base projector.*

Remark 4.1.12. We see through Eq. (4.1) that the base projector has the following left-invariance property:

$$\Pi_{L_a(b)} \circ dL_a = d\sigma_a \circ \Pi_b, \quad a, b \in \mathbb{G}. \tag{4.2}$$

In particular, by putting $b = e$ we obtain

$$\Pi_a \circ dL_a = d\sigma_a \circ \pi_{\mathfrak{k}}, \quad a \in \mathbb{G}. \tag{4.3}$$

So, it is clear that $\operatorname{Ker} \Pi_a = T_a\mathbb{H}$. Take now a left-invariant vector field $Y \in \mathfrak{k}$ and point $a = \mathbf{y} \cdot \mathbf{x}$, $\mathbf{y} \in \mathbb{K}$, $\mathbf{x} \in \mathbb{H}$. Then expanding Eq. (4.3) by the chain rule we obtain that

$$\Pi_a\langle Y(a)\rangle = dL_{\mathbf{y}}(e) \circ dL_{\mathbf{x}}(\mathbf{x}^{-1}) \circ dR_{\mathbf{x}^{-1}}(e)\langle Y(e)\rangle. \tag{4.4}$$

Remark 4.1.13. Note that $\Pi_a = \mathrm{Id}|_{\mathfrak{k}}$ if $a \in \mathbb{K}$. This equality holds for every $a \in \mathbb{G}$ if $\mathbb{G} = \mathbb{K} \times \mathbb{H}$ is a *direct product* of $\mathbb{K}$ and $\mathbb{H}$. Indeed, in this case,

$$dL_{\mathbf{x}}(\mathbf{x}^{-1}) \circ dR_{\mathbf{x}^{-1}}(e)|_{\mathfrak{k}} = d(L_{\mathbf{x}} \circ R_{\mathbf{x}^{-1}})|_{\mathfrak{k}} = \mathrm{Id}|_{\mathfrak{k}}$$

because

$$L_{\mathbf{x}} \circ R_{\mathbf{x}^{-1}}(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y} \cdot \mathbf{x}^{-1} = \mathbf{y} \cdot \mathbf{x} \cdot \mathbf{x}^{-1} = \mathbf{y}.$$

Remark 4.1.14. Using the commutativity of $\pi_{\mathbb{K}}$ with δ_r, we deduce the following invariance of the base operator Π_a w. r. t. dilations:

$$\Pi_{\delta_r(a)} \circ d\delta_r(a) = d\delta_r(\pi_{\mathbb{K}}(a)) \circ \Pi_a, \quad r > 0, \quad a \in \mathbb{G}. \tag{4.5}$$

Proposition 4.1.15. *In compatible coordinates, the action of operator* Π *on* $Y_k \in \mathfrak{k}$ *reads as follows:*

$$\Pi_a\langle Y_k(a)\rangle = \partial_k + \sum_{\deg Y_i > \deg Y_k} P_i^k(\mathbf{y}, \mathbf{x})\partial_i, \tag{4.6}$$

where $P_i^k(\mathbf{y}, \mathbf{x})$ *is a homogeneous polynomial of degree* $\deg P_i^k = \deg Y_i - \deg Y_k$ *meaning that*

$$P_i^k\big(\delta_r(\mathbf{y}), \delta_r(\mathbf{x})\big) = r^{\deg P_i^k} P_i^k(\mathbf{y}, \mathbf{x}).$$

The coefficients of these polynomials P_i^k *depend only on the splitting* $\mathbb{G} = \mathbb{K} \ltimes \mathbb{H}$ *and don't depend on* a *(see Section 4.4 for some concrete examples of their computation).*

Proof. We see from Eq. (4.4) that $\Pi_a\langle Y_k(a)\rangle$ is a left translation, precisely by $dL_{\mathbf{y}}(e) \circ dL_{\mathbf{x}}(\mathbf{x}^{-1})$, of a right invariant vector field $dR_{\mathbf{x}^{-1}}(e)\langle \partial_k \rangle$. So, the claimed form of $\Pi_a\langle Y_k(a)\rangle$ can be derived easily from [FS82, section C, pr. 1.26]. ☐

4.1.4. Graphs in semidirect splitting

Definition 4.1.16. Let $\mathbb{G} = \mathbb{K} \ltimes \mathbb{H}$. A set $\mathcal{S} \subset \mathbb{G}$ is called a $\mathbb{KH}$-*graph* if the projection $\pi_{\mathbb{K}}$ is injective when restricted on $\mathcal{S}$.

Notation 4.1.17. If $\mathcal{S}$ is a $\mathbb{KH}$-graph then it can be represented as

$$\mathcal{S} = \{\Phi(\mathbf{y}) := \mathbf{y} \cdot \phi(\mathbf{y}) \mid \mathbf{y} \in \pi_{\mathbb{K}}(\mathcal{S}) \subset \mathbb{K}\},$$

where $\phi \colon \pi_{\mathbb{K}}(\mathcal{S}) \to \mathbb{H}$. (Thus, $\pi_{\mathbb{K}}(\Phi(\mathbf{y})) = \mathbf{y}$ and $\pi_{\mathbb{H}}(\Phi(\mathbf{y})) = \phi(\mathbf{y})$ for $\Phi(\mathbf{y}) \in \mathcal{S}$.) In this situation, we shall say that ϕ is the *graph-map* for $\mathcal{S}$.

The following properties of $\mathbb{KH}$-graphs are straightforward consequences of the properties of semidirect product in Proposition 4.1.3 (one can find those results in [FSS07, pr. 3.9, 3.10]).

Proposition 4.1.18. *Let* $\mathcal{S}$ *be a* $\mathbb{KH}$*-graph with graph-map* ϕ*.*

- *The dilated set* $\delta_r(\mathcal{S})$*,* $r > 0$*, is also a* $\mathbb{KH}$*-graph with graph-map* $\phi_r := \delta_r \circ \phi \circ \delta_{r^{-1}}$ *defined on* $\delta_r(\pi_{\mathbb{K}}(\mathcal{S})) = \pi_{\mathbb{K}}(\delta_r(\mathcal{S}))$*.*

- *For any $a \in \mathbb{G}$, the left translated set $L_a(\mathcal{S})$ is also a $\mathbb{K}\mathbb{H}$-graph with the graph-map $\phi_a := L_{\pi_{\mathbb{H}}(a)} \circ \phi \circ \sigma_{a^{-1}}$ defined on $\pi_{\mathbb{K}}(L_a(\mathcal{S})) = \sigma_a(\pi_{\mathbb{K}}(\mathcal{S}))$. In particular, if we put $a = \Phi(\mathbf{y}_0)^{-1}$, then*

$$\phi_a(\mathbf{y}) = \phi(\mathbf{y}_0)^{-1} \cdot \phi\big(\mathbf{y}_0 \cdot \phi(\mathbf{y}_0) \cdot \mathbf{y} \cdot \phi(\mathbf{y}_0)^{-1}\big), \quad \phi_a(e) = e.$$

The next result aims at explaining the geometric meaning of the base projector Π.

Proposition 4.1.19. *Let $\mathcal{S}$ be a $\mathbb{K}\mathbb{H}$-graph with a C^1-smooth graph-map $\phi\colon \mathbb{K} \to \mathbb{H}$. Then for every $V \in T_a\mathbb{K}$, $a \in S$, the following relation takes place:*

$$\pi_{\mathfrak{k}}\langle \Pi_a\langle V\rangle \Phi\rangle = V.$$

Let us recall the notations. $\Pi_a\langle V\rangle$ being a vector in $T_{\pi_{\mathbb{K}}(a)}\mathbb{K}$ acts on the function Φ by differentiation and, so that, $d\Phi\langle \Pi_a\langle V\rangle\rangle = \Pi_a\langle V\rangle\Phi \in T_a\mathbb{G}$ and we take the projection of this vector on $T_a\mathbb{K} = dL_a\langle \mathfrak{k}\rangle$.

Proof. If the graph-map ϕ is smooth, then $\mathcal{S}$ is a smooth manifold. Indeed, the parametrization Φ is clearly injective and smooth. The trivial relation $\pi_{\mathbb{K}} \circ \Phi = \mathrm{Id}\,|_{\mathbb{K}}$ implies that $\Pi_a \circ d\Phi(\mathbf{y}) = \mathrm{Id}\,|_{T_{\mathbf{y}}\mathbb{K}}$, $a = \Phi(\mathbf{y})$, so that $d\Phi$ has maximal rank. By the way, because $\operatorname{Ker}\Pi_a = T_a\mathbb{H}$ this also shows that for every $V \in T_a\mathbb{K}$ there is a unique vector $W \in T_a\mathbb{H}$ such that $V + W \in T_a\mathcal{S}$.

Since $\mathcal{S}$ is $\mathbb{K}\mathbb{H}$-graph, $\Phi \circ \pi_{\mathbb{K}}|_{\mathcal{S}} = \mathrm{Id}\,|_{\mathcal{S}}$, therefore, $d\Phi(\mathbf{y}) \circ \Pi_a|_{T_a\mathcal{S}} = \mathrm{Id}\,|_{T_a\mathcal{S}}$. Let $V \in T_a\mathbb{K}$ be arbitrary and $W \in T_a\mathbb{H}$ be such that $W + V \in T_a\mathcal{S}$. Then

$$d\Phi(\mathbf{y}) \circ \Pi_a\langle V\rangle = d\Phi(\mathbf{y}) \circ \Pi_a\langle V + W\rangle = V + W,$$

and, by taking the projection on $T_a\mathbb{K}$ in the last equality we obtain the conclusion. $\square$

4.2. Characterization of Lipschitz graphs

Now we return to metric considerations. Let $\mathbb{G} = \mathbb{K} \ltimes \mathbb{H}$ be a semidirect homogeneous splitting.

Remark 4.2.1. Since $\pi_{\mathbb{K}}$ and $\pi_{\mathbb{H}}$ are homogeneous, $\rho \approx \rho \circ \pi_{\mathbb{K}} + \rho \circ \pi_{\mathbb{H}}$ whatever homogeneous norm ρ we fix on $\mathbb{G}$.

Definition 4.2.2 ([FSC06, Def. 3.1]). A set $\mathcal{S}$ is called a *Lipschitz* $\mathbb{K}\mathbb{H}$-*graph* if there is a constant $0 < C < \infty$ such that

$$\rho \circ \pi_{\mathbb{H}}(a^{-1} \cdot b) \leq C\rho \circ \pi_{\mathbb{K}}(a^{-1} \cdot b) \text{ for all } a, b \in \mathcal{S}. \tag{4.7}$$

The smallest constant $C > 0$ such that this inequality holds is called the Lipschitz constant of $\mathcal{S}$ and denoted by $\mathrm{Lip}(\mathcal{S})$.

We start the study of these geometric objects with a list of simple remarks.

Remark 4.2.3. The notion of Lipschitz $\mathbb{K}\mathbb{H}$-graph does not depend on the choice of homogeneous norm ρ (however, $\mathrm{Lip}(\mathcal{S})$ does).

Remark 4.2.4. If $\mathcal{S}$ is a Lipschitz $\mathbb{KH}$-graph then so is its closure $\bar{\mathcal{S}}$.

Remark 4.2.5. The notion of Lipschitz $\mathbb{KH}$-graph is intrinsic in $\mathbb{G}$. Indeed, if $\mathcal{S}$ is a Lipschitz $\mathbb{KH}$-graph then so are $L_a(\mathcal{S})$, $a \in \mathbb{G}$, and $\delta_r(\mathcal{S})$, $r > 0$, with the same constant $\operatorname{Lip}(\mathcal{S})$.

Remark 4.2.6. It is clear that a Lipschitz $\mathbb{KH}$-graph is indeed a $\mathbb{KH}$-graph. A $\mathbb{KH}$-graph $\mathcal{S}$ with graph-map ϕ is Lipschitz if and only if

$$\rho\big(\phi(\mathbf{y}_1)^{-1} \cdot \phi(\mathbf{y}_2)\big) \leq \operatorname{Lip}(\mathcal{S})\, \rho\big(\phi(\mathbf{y}_1)^{-1} \cdot y_1^{-1} \cdot y_2 \cdot \phi(\mathbf{y}_1)\big), \tag{4.8}$$

for all $\mathbf{y}_1, \mathbf{y}_2 \in \pi_{\mathbb{K}}(\mathcal{S})$. This explicit inequality Eq. (4.8) can be deduced from Eq. (4.1). We note that Eq. (4.8) implies (see Appendix A.5) that

$$\rho\big(\phi(\mathbf{y}_1)^{-1} \cdot \phi(\mathbf{y}_2)\big) \lesssim C\big(\rho(\phi(\mathbf{y}_1))\big) \operatorname{Lip}(\mathcal{S})\, \rho(\mathbf{y}_1^{-1} \cdot \mathbf{y}_2)^{1/\deg \mathbb{G}},$$

In particular, since ϕ is locally bounded, it is also locally Hölder continuous with exponent $1/\deg \mathbb{G}$.

Remark 4.2.7. If $\mathbb{G}$ is the direct product of $\mathbb{K}$ and $\mathbb{H}$, then ϕ is merely a Lipschitz map between $\mathbb{K}$ and $\mathbb{H}$.

Proposition 4.2.8. *Let $\mathcal{S} \subset \mathbb{G}$. A point $a \in \mathcal{S}$ possesses a neighbourhood in which $\mathcal{S}$ is a Lipschitz $\mathbb{KH}$-graph if and only if*

$$\operatorname{pTan}^-_{\mathbb{G}}(\mathcal{S}, a) \cap \mathbb{H} = \{e\}. \tag{4.9}$$

Proof. Recall that the notion of tangent cone is local, so, we can assume that $\mathcal{S}$ is a Lipschitz graph and $v \in \operatorname{pTan}^-_{\mathbb{G}}(\mathcal{S}, a)$ (the largest among tangent cones in Definition 3.2.1). Then by the definition of $\operatorname{pTan}^-_{\mathbb{G}}$ and the homogeneity of the projections $\pi_{\mathbb{K}}$ and $\pi_{\mathbb{H}}$ we easily derive that $\rho(\pi_{\mathbb{H}}(v)) \leq \operatorname{Lip}(\mathcal{S})\rho(\pi_{\mathbb{K}}(v))$. Therefore, if $v \in \mathbb{H}$, i. e. $\pi_{\mathbb{K}}(v) = e$, then $\pi_{\mathbb{H}}(v) = e$.

For the opposite implication, we argue by contradiction. Assume that no neighborhood of a in $\mathcal{S}$ is a Lipschitz graph. This means that there are two sequences of points $\{b_n\}$ and $\{c_n\}$ such that $\lim_n b_n = \lim_n c_n = a$ and $\rho(\pi_{\mathbb{K}}(b_n^{-1} \cdot c_n))/\rho(\pi_{\mathbb{H}}(b_n^{-1} \cdot c_n)) \to 0$. Let v be a limit point of $\delta_{t_n^{-1}}(b_n^{-1} \cdot c_n)$ with $t_n = \rho(b_n^{-1} \cdot c_n) \to 0$. Then $v \in \operatorname{pTan}^-_{\mathbb{G}}(\mathcal{S}, a)$ by definition and $\rho(\pi_{\mathbb{K}}(v))/\rho(\pi_{\mathbb{H}}(v)) = 0$, so that, $v \in \mathbb{H}$. This gives us a contradiction because $\rho(v) = 1$. □

Thus, Proposition 4.2.8 shows that the notion of local Lipschitz $\mathbb{KH}$-graph depends only on $\mathbb{H}$.

Corollary 4.2.9. *Let $\mathcal{S}$ be a compact $\mathbb{KH}$-graph. Then, $\mathcal{S}$ is a Lipschitz $\mathbb{KH}$-graph if and only if Eq. (4.9) holds and its graph-map is continuous.*

Proof. The "only if" implication follows from Proposition 4.2.8 and Remark 4.2.6 in a straightforward manner.

Let us show the opposite implication by contradiction. If $\mathcal{S}$ were not a Lipschitz $\mathbb{KH}$-graph, there should exist two sequences $\{a_n\}_n, \{b_n\}_n \subset \mathcal{S}$ such that

$$\frac{\rho \circ \pi_{\mathbb{H}}(a_n^{-1} \cdot b_n)}{\rho \circ \pi_{\mathbb{K}}(a_n^{-1} \cdot b_n)} \longrightarrow \infty, \quad a_n \neq b_n.$$

In particular, since $\mathcal{S}$ is compact, $\pi_{\mathbb{K}}(a_n^{-1} \cdot b_n) \to e$. The continuity of the graph-map implies that $\pi_{\mathbb{H}}(a_n^{-1} \cdot b_n) \to e$. So, up to extracting a subsequence, there is a point $a \in \mathcal{S}$ such that $\lim a_n = \lim b_n = a$. And this gives us a contradiction with the fact that $\mathcal{S}$ is a Lipschitz $\mathbb{K}\mathbb{H}$-graph in a neighbourhood of a by Proposition 4.2.8. □

Lemma 4.2.10. *If $\mathcal{S}$ is a Lipschitz $\mathbb{K}\mathbb{H}$-graph and $\pi_{\mathbb{K}}(\mathcal{S}) \subset \mathbb{K}$ is open, then $\mathcal{S}$ is locally Ahlfors regular of dimension* $\dim \mathbb{K}$.

Proof. Consider the Borel regular measure $\mu = (\pi_{\mathbb{K}})_{\#} \mathrm{Haar}_{\mathbb{K}}$ on $\mathcal{S}$. Note that μ is invariant under left-translation by Remark 4.1.5. Now we want to show that $\mu(B(a,r) \cap \mathcal{S}) \approx r^{\dim \mathbb{K}}$, $a \in \mathcal{S}$, and use next the standard mass distribution principle. The trick here consists in replacing the homogeneous norm ρ by the equivalent one

$$\bar{\rho} = \mathrm{Lip}(\mathcal{S})\, \rho \circ \pi_{\mathbb{K}} + \rho \circ \pi_{\mathbb{H}},$$

for which $\pi_{\mathbb{K}}(B_{\bar{\rho}}(a,r) \cap \mathcal{S}) = \pi_{\mathbb{K}}(B_{\bar{\rho}}(a,r))$ as soon as $\pi_{\mathbb{K}}(B_{\bar{\rho}}(a,r)) \subset \pi_{\mathbb{K}}(\mathcal{S})$, which holds for $r > 0$ small enough. And now we finish because

$$\mathrm{Haar}_{\mathbb{K}}(\pi_{\mathbb{K}}(B_{\bar{\rho}}(e,r))) = \mathrm{Haar}_{\mathbb{K}}(B_{\bar{\rho}}(e,r) \cap \mathbb{K}) \equiv c r^{\dim \mathbb{K}}.$$

□

Notation 4.2.11. For the rest of the chapter, $\Omega \subset \mathbb{K}$ will denote a relatively open set.

Definition 4.2.12 (projected vector fields)**.** Assume that $\phi\colon \Omega \to \mathbb{H}$ is a graph-map for a Lipschitz $\mathbb{K}\mathbb{H}$-graph. Then for every left-invariant vector field $Y \in \mathfrak{k}$ we define a continuous vector field $\hat{Y}$ on Ω through the action of base projector:

$$\hat{Y}(\mathbf{y}) := \Pi_{\Phi(\mathbf{y})}\langle Y \rangle, \quad \mathbf{y} \in \Omega.$$

Remark 4.2.13. If $\mathbf{y}_1, \mathbf{y}_2 \in \Omega$ and $Y \in \mathfrak{k}$, then by Eq. (4.2)

$$\hat{Y}(\mathbf{y}_2) = d\sigma_{\Phi(\mathbf{y}_1)^{-1} \cdot \Phi(\mathbf{y}_2)}(\mathbf{y}_1)\langle \hat{Y}(\mathbf{y}_1) \rangle.$$

Definition 4.2.14. According to Peano's theorem, for each $Y \in \mathfrak{k}$ and for any $\mathbf{y}_0 \in \Omega$ there is a curve $\gamma \in C^1(I(\gamma), \Omega)$ defined on an open interval $I(\gamma) \ni 0$ such that for $t \in I(\gamma)$

$$\gamma'(t) = \Pi_{\Phi(\gamma(t))}\langle Y \rangle, \quad \gamma(0) = \mathbf{y}_0.$$

We will always suppose that the lifetime interval $I(\gamma)$ for each integral curve γ is maximal. Despite the absence of uniqueness for trajectories of $\hat{Y}$ (that are not Euclidean Lipschitz in general), we introduce the notation $\mathrm{Exp}_{\phi}(tY)(\mathbf{y}_0) := \gamma(t)$ where the choice of a solution γ of the ODE is arbitrary.

The next result presents the behaviour of these integral curves w. r. t. left translations and dilations acting on the graph $\Phi(\Omega)$ in $\mathbb{G}$.

Proposition 4.2.15. *The integral curves* $\mathrm{Exp}_{\phi}(\cdot\, Y)(\mathbf{y})$ *are*

- *left-invariant, i. e.*

$$\sigma_a(\mathrm{Exp}_\phi(tY)(\mathbf{y})) = \mathrm{Exp}_{\phi_a}(tY)(\sigma_a(\mathbf{y})), \quad a \in \mathbb{G}, \quad t \in I(\mathrm{Exp}_\phi(\cdot\, Y)(\mathbf{y})),$$

- *homogeneous, i. e.*

$$\delta_{1/r} \circ \mathrm{Exp}_\phi(r^k tY)(\mathbf{y}) = \mathrm{Exp}_{\phi_r}(tY)(\delta_{1/r}(\mathbf{y})), \quad r > 0, \quad t \in r^{-k} I(\mathrm{Exp}_\phi(\cdot\, Y)(\mathbf{y})),$$

for any homogeneous vector field $Y \in \mathfrak{k} \cap \mathfrak{g}_k$.

Proof. This follows immediately from the corresponding properties of the base projector (see Eq. (4.2) and Eq. (4.5)). □

In the next theorem we give a characterization of $\mathbb{K}\mathbb{H}$-graphs that are locally Lipschitz. We limit our-selves to deal only with local conditions because in the global formulation the implication "2. $\implies$ 3." may fail. Indeed, when we are trying to link points near the boundary of Ω to the inside points by Exp_ϕ trajectories an accessibility issue may appear. To overcome that issue the natural approach would consist in imposing some geometric conditions (*aka* John's domain, see, for instance, [MM05]) on the boundary of Ω. Since it is much more complex we are not going treat this.

Theorem 4.2.16. *Let $\phi\colon \Omega \to \mathbb{H}$ be a continuous map. The following conditions are equivalent:*

1. *$\Phi(\Omega')$ is a Lipschitz $\mathbb{K}\mathbb{H}$-graph for every $\Omega' \Subset \Omega$.*

2. *For any $\Omega' \Subset \Omega$ and for any left-invariant field $Y \in \mathfrak{k} \cap \mathfrak{g}_l$, $l = 1, \dots, \deg \mathbb{K}$, every group valued curve*

$$\begin{aligned} &\Gamma = \Phi \circ \mathrm{Exp}_\phi(\cdot\, Y)(\mathbf{y}), \quad \mathbf{y} \in \Omega', \\ &\Gamma\colon (I(\mathrm{Exp}_\phi(\cdot\, Y)(\mathbf{y})), |\cdot|) \to \Phi(\Omega') \subset (\mathbb{G}, \mathsf{d}), \end{aligned} \tag{4.10}$$

is Hölder continuous with exponent l^{-1} and uniformly bounded Hölder constant provided that $\|Y\| \lesssim 1$.

3. *For any $\Omega' \Subset \Omega$ and for any left-invariant field $Y \in \mathfrak{k} \cap \mathfrak{g}_l$, $l = 1, \dots, \deg \mathbb{K}$, every group valued curve*

$$\phi \circ \mathrm{Exp}_\phi(\cdot\, Y)(\mathbf{y})\colon (I(\mathrm{Exp}_\phi(\cdot\, Y)(\mathbf{y})), |\cdot|) \to \phi(\Omega') \subset (\mathbb{H}, \mathsf{d}_\rho), \quad \mathbf{y} \in \Omega', \tag{4.11}$$

is Hölder continuous with exponent l^{-1} and uniformly bounded Hölder constant provided that $\|Y\| \lesssim 1$.

Notation 4.2.17. Given $\Phi\colon \Omega \to \mathbb{G}$, we denote by $\tilde{\mathsf{d}}(\mathbf{y}, \mathbf{y}') := \mathsf{d}(\Phi(\mathbf{y}), \Phi(\mathbf{y}'))$, $\mathbf{y}, \mathbf{y}' \in \Omega$ the distance induced on Ω by the graph. In the symbol $\tilde{\mathsf{d}}$ we don't emphasize the dependence of the induced distance on Φ because this should be understood from the context.

Proof. There are several implications to be proven.

1. $\Longrightarrow$ **3.** Since all objects are left-invariant, by performing a left translation of $\Phi(\Omega')$ we are free to assume that $\mathbf{y} = e \in \Omega'$, $\phi(e) = e$ and $\Gamma(0) = e$. In this situation, we need to prove that

$$\rho(\phi \circ \gamma(t)) \lesssim \operatorname{Lip}(\Phi(\Omega'))\, |t|^{\frac{1}{\deg Y}}, \quad t \in I(\gamma),$$

where $\gamma(t) = \operatorname{Exp}_\phi(tY)(e)$ is any integral curve of $\hat{Y}$.

We will work in compatible coordinates and to alleviate notation assume that $Y = Y_k$ is one of the basic vector fields. Of course, there is no loss of the generality in this assumption. According to the structure of the vector field $\hat{Y} = \Pi\langle Y \rangle$, see Eq. (4.6), the coordinates of $\gamma(t) = \big(\gamma^1(t), \dots, \gamma^N(t)\big)$ satisfy

$$\begin{cases} \gamma^i(t) \equiv 0, \ \text{if } d_i \le d_k, \ i \neq k, \\ \gamma^k(t) = t, \\ \gamma^i(t) = \displaystyle\int_0^t P^i_k\big(\gamma(s), \phi \circ \gamma(s)\big)\, ds, \ \text{if } d_i > d_k. \end{cases}$$

Here, we introduced a shorter notation for $\deg Y_i = d_i$.

By homogeneity,

$$|P^i_k(\mathbf{y}, \mathbf{x})| \le C\|Y\|\big(\rho(\mathbf{x})^{d_i - d_k} + \rho(\mathbf{y})^{d_i - d_k}\big),$$

with some universal constant $C > 0$. Let us only consider a positive range of values of t and introduce two maximal functions

$$m_\gamma(t) = \max_{s \in [0,t]} \rho(\gamma(s)),$$
$$m_\phi(t) = \max_{s \in [0,t]} \rho(\phi \circ \gamma(s)).$$

So, using the homogeneity of P^i_k we can write a very naive estimate

$$\rho(\gamma(t)) \lesssim \max \big(\gamma^i(t)\big)^{\frac{1}{d_i}} \lesssim \max_{d_i > d_k} \left\{ t^{\frac{1}{d_k}},\ t^{\frac{1}{d_i}} m_\gamma(t)^{1 - \frac{d_k}{d_i}},\ t^{\frac{1}{d_i}} m_\phi(t)^{1 - \frac{d_k}{d_i}} \right\}.$$

By taking the maximum of its left-hand side we arrive to the following inequality:

$$m_\gamma(t) \lesssim \max_{d_i > d_k} \left\{ t^{\frac{1}{d_k}},\ t^{\frac{1}{d_i}} m_\gamma(t)^{1 - \frac{d_k}{d_i}},\ t^{\frac{1}{d_i}} m_\phi(t)^{1 - \frac{d_k}{d_i}} \right\}.$$

And by solving it w. r. t. m_γ we obtain

$$m_\gamma(t) \lesssim \max_{d_i > d_k} \left\{ t^{\frac{1}{d_k}},\ t^{\frac{1}{d_i}} m_\phi(t)^{1 - \frac{d_k}{d_i}} \right\}. \tag{4.12}$$

It is time to use assumption **13.**, i. e. $\rho(\phi \circ \gamma(t)) \le \operatorname{Lip}(\Phi(\Omega'))\, \rho(\gamma(t))$. Passing to maximal functions, we derive

$$m_\phi(t) \lesssim C m_\gamma(t) \lesssim C \max_{d_i > d_k} \left\{ t^{\frac{1}{d_k}},\ t^{\frac{1}{d_i}} m_\phi(t)^{1 - \frac{d_k}{d_i}} \right\}, \quad C = \operatorname{Lip}(\Phi(\Omega')).$$

Again, by solving this inequality we get $m_\phi(t) \lesssim C t^{\frac{1}{d_k}}$, and, thus, we proved **3.**.

3. $\iff$ **2.** It is obvious that **2.** is stronger than **3.** because $\pi_{\mathbb{H}}$ is a Carnot group homomorphism. If **3.** is verified, then the maximal function m_ϕ satisfies $m_\phi(t) \lesssim C|t|^{\frac{1}{d_k}}$. Therefore, using Eq. (4.12) we obtain that $\rho(\gamma(t)) \lesssim C|t|^{\frac{1}{d_k}}$ and, hence,

$$\rho(\Phi \circ \gamma(t)) \approx \rho(\gamma(t)) + \rho(\phi \circ \gamma(t)) \lesssim C|t|^{\frac{1}{d_k}}.$$

2. $\Longrightarrow$ **1.** For two points $a, b \in \Phi(\Omega')$ we need to prove that

$$\rho \circ \pi_{\mathbb{H}}(a^{-1} \cdot b) \lesssim_{\Omega'} \rho \circ \pi_{\mathbb{K}}(a^{-1} \cdot b).$$

Using left-invariance, we can always assume that $a = e$, so $\phi(e) = e \in \Omega'$, in which case we have to prove that $\rho\big(\phi(\mathbf{y})\big) \lesssim \rho(\mathbf{y})$ for $\mathbf{y} \in \Omega'$.

For a fixed compatible basis $\{Y_k\}_{i=1,\dots,N}$ on $\mathfrak{k}$ let us consider the set of points $\Omega_{\mathrm{Exp}_\phi}(\mathbf{y})$ admitting a representation

$$\Omega \ni \mathbf{y}' = \mathrm{Exp}_\phi(t_N Y_N) \circ \ldots \circ \mathrm{Exp}_\phi(t_1 Y_1)(\mathbf{y}), \quad (t_1, \dots t_N) \in \mathbb{R}^N. \tag{4.13}$$

For $\mathbf{y}' \in \Omega_{\mathrm{Exp}_\phi}(\mathbf{y})$ we denote by $\Omega_\phi(\mathbf{y}, \mathbf{y}') \subset \Omega$ the union of all trajectories $\{\mathrm{Exp}_\phi(sY_i)(\mathbf{y}_{i-1}) \mid s \in [0, t_i]\}$, $i = 1, \dots, N$. It is a compact set. Note that because the matrix of vector fields $\{\hat{Y}_i\}_{i=1,\dots,N}$ is unipotent, the set $\Omega_{\mathrm{Exp}_\phi}(\mathbf{y})$ contains some open neighbourhood of $\mathbf{y}$. We can view the tuple $(t_1, \dots, t_N)$ as "second kind" coordinates (see [Var74, p. 81], [FS82, Lemma 1.31]) on $\mathbb{K}$ w. r. t. to vector fields $\{\hat{Y}_i\}_{i=1,\dots,N}$ except the fact that this tuple representation is not necessarily unique due to the possible lack of uniqueness for trajectories Exp_ϕ.

For $\mathbf{y} \in \Omega_{\mathrm{Exp}_\phi}(e)$ we introduce a chain of points $\{e = \mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_N = \mathbf{y}\}$ with $\mathbf{y}_i = \mathrm{Exp}_\phi(t_i Y_i)(\mathbf{y}_{i-1})$, see Figure 4.1. Due to the form of $\hat{Y}_i$, the following relations between the coordinates of $\mathbf{y}_i$ and $\{t_i\}$ hold:

$$\begin{aligned} t_{i+1} &= \mathbf{y}^{i+1} - \mathbf{y}_i^{i+1}; \\ \mathbf{y}_i^k &= \mathbf{y}^k \text{ if } 1 \leq k \leq i. \end{aligned} \tag{4.14}$$

(Upper inscriptions are used for the coordinates index.) This allows us to effectively find $(t_1, \dots, t_N)$ step by step.

Let the constant $K = K(\Omega(e, \mathbf{y}))$ be an upper bound for the Hölder constants of $\Phi \circ \mathrm{Exp}_\phi$ for trajectories Exp_ϕ from $\Omega(e, \mathbf{y})$. Our goal now is to show that $\rho(\mathbf{y}_k) \lesssim_K \rho(\mathbf{y})$ for $k = 0, \dots, N-1$. Indeed, this will imply that $|t_k| \lesssim_K \rho(\mathbf{y})^{d_k}$ for $k = 1, \dots, N$, and so, by our hypothesis and the triangle inequality, will give us that $\rho\big(\phi(\mathbf{y})\big) \lesssim_K \rho(\mathbf{y})$.

We proceed by induction over k. For $k = 0$, the statement is obvious because $\mathbf{y}_0 = e$. Let us assume that the statement holds for some $k > 0$ and proceed to prove it for $k+1$. First of all, we note that

$$|t_{k+1}| = |\mathbf{y}^{k+1} - \mathbf{y}_k^{k+1}| \leq |\mathbf{y}^{k+1}| + |\mathbf{y}_k^{k+1}| \lesssim \rho(\mathbf{y})^{d_k} + \rho(\mathbf{y}_k)^{d_k} \lesssim_K \rho(\mathbf{y})^{d_k},$$

by the induction hypothesis. Let $\gamma(t) := \mathrm{Exp}_\phi(tY_{k+1})(\mathbf{y}_k)$, $t \in [0, t_{k+1}]$. Then $\gamma^i(t) \equiv \mathbf{y}^i$ if $i < k+1$, so there is nothing to do for those coordinates. We will now prove by

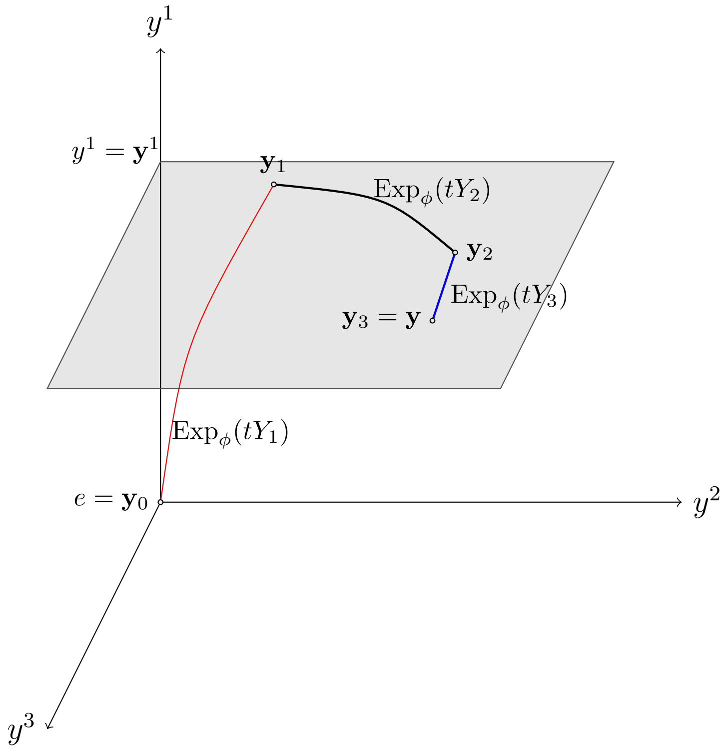

Figure 4.1.: Chain of points $\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ in $\mathbb{K}$ for $\dim \mathbb{K} = 3$

induction over $d_i \geq d_{k+1}$ that $|\gamma^i(t)| \lesssim_K \rho(\mathbf{y})^{d_i}$. Since among the coordinates of degree d_{k+1}, only the $(k+1)$-th coordinate is changed:

$$\gamma^{k+1}(t) = t + \mathbf{y}_k^{k+1},$$

therefore, the induction basis is verified. For the induction step, assuming that $|\gamma^j(t)| \lesssim_K \rho(\mathbf{y})^{d_j}$ for all j such that $d_{k+1} \leq d_j < d_i$, we need to show that $|\gamma^i(t)| \lesssim_K \rho(\mathbf{y})^{d_i}$ for $t \in [0, t_{k+1}]$. These coordinates satisfy

$$\gamma^i(t) = \mathbf{y}_k^i + \int_0^t P_{k+1}^i\big(\gamma(s), \phi \circ \gamma(s)\big)\, ds.$$

By our main assumption, $\rho(\phi \circ \gamma(s), \phi \circ \gamma(0)) \leq K|s|^{1/d_{k+1}}$. By the first induction assumption, we get that $\rho(\phi \circ \gamma(0)) = \rho(\phi(\mathbf{y}_k)) \lesssim_K \rho(\mathbf{y})$, so that, by the triangle inequality, $\rho(\phi \circ \gamma(s)) \lesssim_K \rho(\mathbf{y}) + |s|^{1/d_{k+1}}$.

The polynomial P_{k+1}^i is homogeneous of degree $d_i - d_{k+1} < d_i$, so it may effectively depend only on the coordinates of degree strictly less than d_i. With this remark, using the second induction assumption we obtain that

$$
\begin{aligned}
|\gamma^i(t)| &\le |\mathbf{y}_k^i| + \int_0^t |P_{k+1}^i(\gamma(s), \phi\circ\gamma(s))|\, ds \\
&\lesssim |\mathbf{y}_k^i| + \int_0^{|t_{k+1}|} \rho\big(\{\gamma^j(s)\}_{d_j<d_i}\big)^{d_i-d_{k+1}} + \rho\big(\phi\circ\gamma(s)\big)^{d_i-d_{k+1}}\, ds \\
&\lesssim_K \rho(\mathbf{y})^{d_i} + \int_0^{|t_{k+1}|} \rho(\mathbf{y})^{d_i-d_{k+1}} + |s|^{\frac{d_i}{d_{k+1}}-1}\, ds \\
&\lesssim \rho(\mathbf{y})^{d_i} + |t_{k+1}|\rho(\mathbf{y})^{d_i-d_{k+1}} + |t_{k+1}|^{\frac{d_i}{d_{k+1}}} \\
&\lesssim_K \rho(\mathbf{y})^{d_i} + \rho(\mathbf{y})^{d_{k+1}}\rho(\mathbf{y})^{d_i-d_{k+1}} + \rho(\mathbf{y})^{d_{k+1}\frac{d_i}{d_{k+1}}} \\
&\lesssim \rho(\mathbf{y})^{d_i}.
\end{aligned}
$$

This accomplishes the second induction over d_i and, as a consequence, the first induction over k as well.

Thus, we obtain a local version of Lipschitz graph inequality Eq. (4.7), i. e. it is valid for points $a = \Phi(\mathbf{y})$ and $b = \Phi(\mathbf{y}')$ such that $\mathbf{y}, \mathbf{y}' \in \Omega$ and $\mathbf{y}' \in \Omega_{\mathrm{Exp}_\phi}(\mathbf{y})$. The constant $C = C(\mathbf{y}, \mathbf{y}')$ here is uniformly bounded provided the distance from the trajectory set $\Omega(\mathbf{y}, \mathbf{y}')$ to the boundary of Ω stays uniformly bounded from below.

Our argument shows also that the size of the trajectory set $\mathrm{diam}_{\tilde{\mathsf{d}}}(\Omega(\mathbf{y}, \mathbf{y}')) \eqsim_C \tilde{\mathsf{d}}(\mathbf{y}, \mathbf{y}')$. The only obstruction for $\mathbf{y}' \in \Omega$ to belong to $\Omega_{\mathrm{Exp}_\phi}(\mathbf{y})$ is that in the recurrent definition of trajectories $\{\mathrm{Exp}_\phi(t_i Y_i)\}$ by Eq. (4.14) one of these trajectories "reaches" the boundary of Ω (or more correctly, leaves any compact of Ω). Therefore, if $\Omega' \Subset \Omega'' \Subset \Omega$ are fixed and $R = \tilde{\mathsf{d}}(\partial\Omega', \partial\Omega'') > 0$, then there is $r \eqsim_C R$, $C = C(\Omega'')$, such that

$$
\begin{aligned}
\{\mathbf{y}' \in \Omega' \mid \tilde{\mathsf{d}}(\mathbf{y}, \mathbf{y}') < r\} &\subset \Omega_{\mathrm{Exp}_\phi}(\mathbf{y}), \quad \mathbf{y} \in \Omega', \\
\Omega(\mathbf{y}, \mathbf{y}') &\subset \Omega'' \;\text{ if }\; \tilde{\mathsf{d}}(\mathbf{y}, \mathbf{y}') < r, \quad \mathbf{y}, \mathbf{y}' \in \Omega'.
\end{aligned}
$$

To conclude, take $\mathbf{y}, \mathbf{y}' \in \Omega'$. If $\tilde{\mathsf{d}}(\mathbf{y}, \mathbf{y}') < r$, then $\mathbf{y}' \in \Omega'(\mathbf{y})$, we can use a local estimate for Lipschitz $\mathbb{KH}$-graphs with Lipschitz constant $C = C(\Omega'')$. Otherwise, i. e. $\tilde{\mathsf{d}}(\mathbf{y}, \mathbf{y}') \ge r$, we will perform a naive estimate using the boundedness of ϕ on Ω'. □

Remark 4.2.18. Observe that $\rho(\Gamma(t)) \gtrsim \rho(\gamma(t)) \gtrsim |t|^{\frac{1}{d_k}}$ in the proof of Theorem 4.2.16 due to the presence of the explicit term $\gamma^k(t) = t$. This means that Γ is in fact a bi-Hölder curve of exponent $1/\deg Y$.

Remark 4.2.19. Note that under the assumptions of Theorem 4.2.16, if Eq. (4.13) holds for $\mathbf{y}, \mathbf{y}' \in \Omega$ then

$$\tilde{\mathsf{d}}(\mathbf{y}, \mathbf{y}') \eqsim_K \max_{k=1,\dots,N} |t_k|^{1/d_k}$$

where the equivalence constant K is a function of $\mathrm{Lip}(\Phi(\Omega(\mathbf{y}, \mathbf{y}')))$.

Remark 4.2.20. This remark is about the possible non-uniqueness of the trajectories $\mathrm{Exp}_\phi(\cdot Y)(\mathbf{y})$ in Theorem 4.2.16 (or in Theorem 4.3.1 below). Note that in the implication "**1.** $\implies$ **3.**" we prove in fact that Eq. (4.11) holds for *any* integral curve $\mathrm{Exp}_\phi(\cdot Y)(\mathbf{y})$ and *any* $Y \in \mathfrak{k}$. Meanwhile, for the inverse implication we only need that for every $\mathbf{y} \in \Omega$, Eq. (4.11) holds for *some* integral curve $\mathrm{Exp}_\phi(\cdot Y)(\mathbf{y})$ and, moreover, only for vector fields Y that belong to $\{Y_k\}_{k=1,\dots,N}$ *some* fixed in advance basis of $\mathfrak{k}$. The proof of "**3.** $\iff$ **2.**" establishes the equivalence between Eq. (4.10) and Eq. (4.11) for the same choice of the integral curve $\mathrm{Exp}_\phi(\cdot Y)(\mathbf{y})$.

4.3. Characterization of regular surfaces

Theorem 4.3.1. *Assume that $\mathbb{G} = \mathbb{K} \ltimes \mathbb{H}$ where $\mathbb{H}$ is horizontal and $\mathbb{K}$ is vertical. Let $\mathcal{S}$ be a $\mathbb{K}\mathbb{H}$-graph with a continuous graph-map $\phi\colon \Omega \to \mathbb{H}$. The following conditions are equivalent:*

1. *$\mathcal{S} \subset F^{-1}(e)$ is a co-abelian surface of codimension $\dim \mathbb{H}$ with $F \in C^1_h(\mathbb{G}, \mathbb{H})$ such that*

$$\mathrm{Ker}\, D_h F(a) \cap \mathbb{H} = \{e\}, \quad a \in \mathcal{S}. \tag{4.15}$$

2. *For any $Y \in \mathfrak{k}$ and any $\mathbf{y} \in \Omega$ we define an integral curve $\gamma(t) := \mathrm{Exp}_\phi(tY)(\mathbf{y})$. Depending on $\deg Y$ one of two statements below holds for $\phi \circ \gamma$.*

A. $\deg Y = 1$*: There exists a linear map*

$$w_{\mathbf{y}} \colon \mathfrak{k} \cap \mathfrak{g}_1 \to \mathfrak{h},$$

continuously depending on $\mathbf{y} \in \Omega$ such that

$$\frac{d}{dt}(\phi \circ \gamma)(t) = w_{\gamma(t)}\langle Y \rangle, \quad t \in I(\gamma). \tag{4.16}$$

B. $\deg Y \geq 2$*: Then*

$$\phi \circ \gamma \in \mathrm{hol}^{\frac{1}{\deg(Y)}}(I, \mathbb{H}), \quad \mathbf{y} \in \Omega, \quad I \Subset I(\gamma), \tag{4.17}$$

where the small-o in the definition of Hölder class is uniform (see Definition 2.1.12) provided that $\|Y\| \lesssim 1$ and $\gamma(I) \subset \Omega'$ for some set $\Omega' \Subset \Omega$ fixed in advance.

Proof. We must prove two implications.

1. $\implies$ **2.** First, we observe that $\Phi(\Omega')$ is a Lipschitz $\mathbb{K}\mathbb{H}$-graph for any $\Omega' \Subset \Omega$. Indeed, by Theorem 3.3.3, $\mathrm{pTan}^-_{\mathbb{G}}(\mathcal{S}, a) \subset \mathrm{pTan}^-_{\mathbb{G}}(F^{-1}(e), a) = \mathrm{Ker}\, D_h F(a)$, $a \in \Phi(\Omega)$. By our assumption, $\mathrm{pTan}^-_{\mathbb{G}}(\mathcal{S}, a) \cap \mathbb{H} = \{e\}$ and ϕ is continuous, so we can apply Corollary 4.2.9. Therefore, by Theorem 4.2.16 the composition $\phi \circ \gamma \in \mathrm{Hol}^{1/\deg Y}(I, \mathbb{H})$ where $I \Subset I(\gamma)$ such that $\gamma(I) \subset \Omega'$.

Case A: $\deg Y = 1$. If $\deg Y = 1$, then $\phi \circ \gamma$ is a locally Lipschitz curve. In particular, it is differentiable for almost all points $t \in I(\gamma)$. Let us take a point $t \in I(\gamma)$ of differentiability of $\phi \circ \gamma$. Then t is obviously also a differentiability point of $\Phi \circ \gamma$ (because $\gamma \in C^1$). So, let us compute the differential of $\Phi \circ \gamma$ at point t. Since everything is left-invariant, we can assume for simplicity that $\phi(e) = e$ and $\gamma(t) = e$. Using the fact that $\Phi(\mathbf{y}) = L_{\mathbf{y}}(\phi(\mathbf{y})) = R_{\phi(\mathbf{y})}(\mathbf{y})$ we obtain that

$$\begin{aligned} \frac{d}{dt}(\Phi \circ \gamma)(t) &= dL_{\gamma(t)}\langle(\phi \circ \gamma)'(t)\rangle + dR_{\phi\circ\gamma(t)}\langle\gamma'(t)\rangle \\ &= dL_e\langle(\phi \circ \gamma)'(t)\rangle + dR_e\langle\Pi_e\langle Y\rangle\rangle = (\phi \circ \gamma)'(t) + Y \in (\mathfrak{h} + \mathfrak{k}) \cap \mathfrak{g}_1. \end{aligned}$$

By assumption, F is horizontally differentiable, hence we can apply the chain rule to compute the derivative of the composition $F \circ \Phi \circ \gamma \equiv \mathit{const}$ at t. This results in the following equation:

$$d_h F(e)\langle(\phi \circ \gamma)'(t) + Y\rangle = 0, \tag{4.18}$$

where $d_h F(e) = \log \circ D_h F(e) \circ \exp$ is the horizontal differential of F written on the Lie algebra level.

Note that Eq. (4.15) ensures that $d_h F(a) \llcorner \mathfrak{h}$ is a horizontal automorphism of $\mathfrak{h}$ for any $a \in \mathcal{S}$. Let us denote by $T_a \colon \mathfrak{h} \to \mathfrak{h}$ the inverse of $d_h F(a) \llcorner \mathfrak{h}$, so that $T_a \circ d_h F(a)\langle X\rangle = X$ for all $X \in \mathfrak{h}$. It is clear that T_a depends continuously on $a \in \mathcal{S}$ because $d_h F(a)$ does. Using this we derive from Eq. (4.18) that $(\phi \circ \gamma)'(t) = -(T_{\Phi(\gamma(t))} \circ d_h F(a))\langle Y\rangle$. We see that this derivative is continuous in $t \in I(\gamma)$, so that, $\phi \circ \gamma \in C^1(I(\gamma), \mathbb{H})$. Furthermore, because $(\phi \circ \gamma)'(t)$ is horizontal, the C^1 curves $\phi \circ \gamma$ and, therefore, $\Phi \circ \gamma$ are horizontal curves in $\mathbb{H}$ and $\mathbb{G}$ respectively. Thus, we have completed **2.A** with $w_{\mathbf{y}} = -T_{\Phi(\mathbf{y})} \circ d_h F(\Phi(\mathbf{y})) \llcorner \mathfrak{k} \cap \mathfrak{g}_1$.

Case B: $\deg Y \geq 2$. Fix an open set $\Omega' \Subset \Omega$. Take two points $t_0, t \in I(\gamma)$ such $\gamma([t_0, t]) \subset \Omega'$. Of course, this is only possible if $\gamma(I(\gamma)) \cap \Omega' \neq \varnothing$. Again by left-invariance we assume that $\Phi(\gamma(t_0)) = e$. Since points $\Phi(\gamma(t_0))$ and $\Phi(\gamma(t))$ lie on $F^{-1}(e)$, the definition of the horizontal differentiability of F reads

$$\rho\big(D_h F(e)\langle\Phi \circ \gamma(t)\rangle\big) = o\big(\rho(\Phi \circ \gamma(t))\big), \quad t \to t_0,$$

where the small-o is uniform. Note that because $\deg Y \geq 2$, according to the structure of the vector field $\hat{Y}$, the integral curve $\gamma(t)$ has no horizontal components: $\pi_{\mathfrak{g}_1}\langle\log \circ \gamma(t)\rangle = 0$. Therefore,

$$D_h F(e)\langle\Phi \circ \gamma(t)\rangle = D_h F(e)\langle\gamma(t)\rangle \cdot D_h F(e)\langle\phi \circ \gamma(t)\rangle = D_h F(e)\langle\phi \circ \gamma(t)\rangle,$$

because the target space $\mathbb{H}$ has degree 1. Thanks to Eq. (4.15), we obtain that

$$\rho(D_h F(a)\langle h\rangle) \gtrsim \rho(h), \quad h \in \mathbb{H}, \quad a \in \Phi(\Omega').$$

By Hölder continuity, $\rho(\Phi \circ \gamma(t)) \lesssim |t - t_0|^{1/\deg Y}$. Thus, we conclude by

$$\rho(\phi \circ \gamma(t)) \lesssim \rho(D_h F(a)\langle\Phi \circ \gamma(t)\rangle) = o(\rho(\Phi \circ \gamma(t))) = o(|t - t_0|^{1/\deg Y}).$$

2. $\implies$ **1.** To prove this implication we are going to verify that $\mathcal{S}$ satisfies the assumptions of Theorem 3.1.12. A natural candidate for the tangent space W_a to $\mathcal{S}$ at point $a \in \mathcal{S}$ is the $\mathbb{K}\mathbb{H}$-graph with graph-map

$$\psi_a := \exp \circ w_{\pi_{\mathbb{K}}(a)} \circ \pi_{\mathfrak{k}\cap\mathfrak{g}_1} \circ \log . \tag{4.19}$$

In order to show that W_a is a vertical (and, hence, homogeneous) subgroup of $\mathbb{G}$, we note that W_a is the kernel of some homogeneous homomorphism $q\colon \mathbb{G} \to \mathbb{H}$. We define q at Lie algebra level, i. e. a linear map from $\mathfrak{g}_1$ to $\mathfrak{h}$, as follows:

$$X + Y \to X - w_{\pi_{\mathbb{K}}(a)}\langle Y\rangle, \quad Y \in \mathfrak{k}\cap\mathfrak{g}_1, \quad X \in \mathfrak{h}.$$

Any element $b \in W_a$ can be written in a unique way as $b = \exp(Y + Y') \cdot \exp(w_{\pi_{\mathbb{K}}(a)}\langle Y\rangle)$ where $Y \in \mathfrak{k}\cap\mathfrak{g}_1$, $Y' \in \mathfrak{k}$ and $\pi_{\mathfrak{g}_1}(Y') = 0$. Then, indeed, we check that $\operatorname{Ker} q = W_a$ by

$$\begin{aligned} q(b) &= q(\exp(Y + Y')) \cdot q(\exp(w_{\pi_{\mathbb{K}}(a)}\langle Y\rangle)) \\ &= \exp(-w_{\pi_{\mathbb{K}}(a)}\langle Y\rangle) \cdot \exp(w_{\pi_{\mathbb{K}}(a)}\langle Y\rangle) \\ &= \exp(-w_{\pi_{\mathbb{K}}(a)}\langle Y\rangle + w_{\pi_{\mathbb{K}}(a)}\langle Y\rangle) = e. \end{aligned}$$

We also don't forget to mention that by definition $W_a \cap \mathbb{H} = \{e\}$.

Fix $\Omega' \Subset \Omega$. Note that our assumptions are stronger that those of Theorem 4.2.16, so that $\mathcal{S}$ is a local Lipschitz $\mathbb{K}\mathbb{H}$-graph. So, any point $a \in \mathcal{S}' = \Phi(\Omega')$ has a neighbourhood $B(a, r)$ with some fixed radius $r \approx \tilde{\mathsf{d}}(\partial\Omega', \partial\Omega) > 0$ such that $\mathcal{S} \cap B(a, r)$ is a Lipschitz $\mathbb{K}\mathbb{H}$-graph. As usual, by left-translating we may assume that $a = e$. In this situation, since both $\mathcal{S}$ and W_a are $\mathbb{K}\mathbb{H}$-graphs, in order to prove the Reifenberg flatness property it is enough to show that

$$\rho(\phi(\mathbf{y})^{-1} \cdot \psi_a(\mathbf{y})) = o(\rho(\Phi(\mathbf{y}))), \quad \Omega \ni \mathbf{y} \to e, \tag{4.20}$$

where the small-o is uniform. To show this uniformity, we may consider only points $\mathbf{y} \in B_{\tilde{\mathsf{d}}}(e, r)$ because the radius r is uniformly bounded from below. For such points $\mathbf{y}$ we can replace $\rho(\Phi(\mathbf{y}))$ by $\rho(\mathbf{y})$ in Eq. (4.20) loosing only the constant $\operatorname{Lip}(\mathcal{S} \cap B(a, r))$.

Let us reuse the notations from the proof of Theorem 4.2.16. We can assume that the value of $r > 0$ is small enough to state that $B_{\tilde{\mathsf{d}}}(e, r) \subset \Omega_\phi(e)$, i. e. any $\mathbf{y} \in B_{\tilde{\mathsf{d}}}(e, r)$ admits a representation

$$\mathbf{y} = \operatorname{Exp}_\phi(t_N Y_N) \circ \ldots \circ \operatorname{Exp}_\phi(t_1 Y_1)(e).$$

According to Eqs. (4.14) and (4.19), in compatible coordinates, $\psi_a(\mathbf{y})$ reads as follows:

$$\psi_a(\mathbf{y}) = w_e\langle t_1 Y_1 + \ldots + t_{n_1} Y_{n_1}\rangle,$$

where $\{Y_k\}_{1,\ldots,n_1}$ is a basis of $\mathfrak{k}\cap\mathfrak{g}_1$. By Eq. (4.16) and the Fundamental theorem of calculus

$$\phi(\mathbf{y}_k) = \phi(\mathbf{y}_{k-1}) + \int_0^{t_k} w_{\operatorname{Exp}_\phi(sY_k)(\mathbf{y}_{k-1})}\langle Y_k\rangle \;\; = \phi(\mathbf{y}_{k-1}) + t_k w_e\langle Y_k\rangle + t_k \alpha(t_k),$$

where

$$|\alpha(t_k)| \leq 2 \max_{s\in[0,t_k]} |w_e\langle Y_k\rangle - w_{\mathrm{Exp}_\phi(sY_k)(\mathbf{y}_{k-1})}\langle Y_k\rangle| \to 0, \quad t_k \to 0.$$

Applying this recursively we obtain that

$$\begin{aligned}\phi(\mathbf{y}_{n_1}) &= t_1 w_e\langle Y_1\rangle + \ldots + t_{n_1} w_e\langle Y_{n_1}\rangle + o(|t_1| + \ldots + |t_{n_1}|)\\ &= \psi_a(\mathbf{y}) + o(|t_1| + \ldots + |t_{n_1}|),\end{aligned}$$

where the small-o is uniform. Next, for $d_k \geq 2$ Eq. (4.17) gives a uniform estimate

$$\rho(\phi(\mathbf{y}_{k-1})^{-1} \cdot \phi(\mathbf{y}_k)) = o(|t_k|^{1/d_k}).$$

Thus, by the triangle inequality,

$$\rho(\psi_a(\mathbf{y})^{-1} \cdot \phi(\mathbf{y})) = o(\max_{k=1,\ldots,N} |t_k|^{1/d_k}),$$

Now we can conclude because by Theorem 4.2.16, $\rho(\mathbf{y}) \approx \max_{k=1,\ldots,N} |t_k|^{1/d_k}$. □

Let us now make some comments about Theorem 4.3.1. As in the previous characterizations of regular surfaces (like Theorem 3.3.5), we are limited to consider only the abelian group $\mathbb{H} \simeq \mathbb{R}^M$ because the implication "**2.** ⇒ **1.**" requires the Whitney extension theorem. However, for a general subgroup $\mathbb{H}$, what remains true is a characterization of a locally Reifenberg flat set $\mathcal{S} \subset \mathbb{G}$ with tangents W_a that are complementary to $\mathbb{H} \subset \mathbb{G}$, The family of these locally Reifenberg flat sets includes level sets of C^1_h maps, but it can also be larger as we already observed in Section 3.1.4.

Note that we don't need any approximation of $\mathcal{S}$ by a smooth surface but derive our results (such as Eq. (4.16)) directly using only metric considerations. That is one more reason for such result to be interesting for non-abelian $\mathbb{H}$.

Remark 4.3.2. If ϕ is C^1 in a classical sense, then the trajectories $\mathrm{Exp}_\phi(tY)(\mathbf{y})$ are unique and C^1 depend on t and $\mathbf{y}$. It is obvious that condition **2.B** always holds for such ϕ and condition **2.A** merely means that (compare with Proposition 4.1.19)

$$\Pi_{\Phi(\mathbf{y})}\langle Y\rangle\phi(\mathbf{y}) = w_{\mathbf{y}}\langle Y\rangle, \quad Y \in \mathfrak{k} \cap \mathfrak{g}_1. \tag{4.21}$$

In particular, a $\mathbb{KH}$-graph with a C^1 graph-map is always a regular surface.

In which sense can we understand Eq. (4.21) for maps ϕ that are not C^1 but merely C^0? One possibility can be to declare that a continuous function $\phi\colon \Omega \to \mathbb{H}$ solves Eq. (4.21) *along characteristics* if Eq. (4.16) is satisfied, see [BS10b; ASV06]. Indeed, it does make a senses because for a continuous ϕ we can define *characteristics* $\mathrm{Exp}_\phi(tY)(\mathbf{y})$ and to look at the metric behaviour of ϕ along it. To handle the non-uniqueness of the trajectories we can, for instance, either require that Eq. (4.16) holds for all characteristics either that for every point $\mathbf{y}$ there is some characteristic starting at $\mathbf{y}$ such that Eq. (4.16) holds.

For a scalar function ϕ (when $\mathbb{H} \equiv \mathbb{R}$) there is another possibility: Eq. (4.21) can be also understood in a weak sense. Let us illustrate this with an example. Consider a scalar function $\phi = \phi(v, y, z)\colon \Omega \to \mathbb{R}$ describing an X-graph in Engel group E^4 (see Section 4.4.1). Here, we obtain one quasi-linear equation (since $\dim \mathfrak{k} \cap \mathfrak{g}_1 = 1$)

$$(\partial_v - \phi\partial_y - \frac{y + v\phi}{2}\partial_z)\phi = \partial_v\phi - \frac{1}{2}\partial_y\phi^2 - \frac{y}{2}\partial_z\phi - \frac{v}{4}\partial_z\phi^2 = w(v, y, z).$$

Take now a test function $\psi \in C^1(\Omega)$ with compact support $\mathrm{supp}(\psi) \Subset \Omega$. As usual, by multiplying both sides with ψ and integrating by parts, we can put all partial derivatives on ψ:

$$-\int_\Omega w\psi = \int_\Omega \phi\partial_v\psi - \frac{\phi^2}{2}\partial_y\psi - \frac{y\phi}{2}\partial_z\psi - \frac{v\phi^2}{4}\partial_z\psi. \tag{4.22}$$

Now we see that it is problematic to perform the same trick for higher dimensional $\mathbb{H}$ because the mixed terms like $\ldots + \phi^1\partial_x\phi^2 + \ldots$ can appear and it is not clear how to handle them (see Remark 4.6.2). We think that in general these weak formulations are only possible for the sets of codimension one because they rise as boundary of open sets in $\mathbb{G}$ for which the notion of the finite perimeter preexists and naturally provides a weak formulation.

Of course, Eq. (4.22) makes sense not only for $\phi \in C^1$ but also for $\phi \in L_{1,loc}(\Omega)$. So, one may wish to get such a formulation for a continuous function ϕ describing a regular surface. The standard way to show Eq. (4.22) is to find a smooth approximation $\phi_n \in C^1(\Omega, \mathbb{R})$ of ϕ such that $\phi_n \to \phi$ and $\Pi_{\Phi_n(\mathbf{y})}\langle Y\rangle\phi_n \to w$ locally uniformly on Ω. Yet, an issue here is that the standard *aka* convolutional approximation $\phi_n = \phi \star \eta_n$ may fail to accomplish this because it does not take into account the non-linear structure of the equation. Nevertheless, in this situation the convenient approximation can be found using the exterior geometric approach. Mainly, we recall that the surface described by ϕ is a level set of some map $F \in C^1_h$. We should first approximate F by smooth mappings F_n such that $F_h \to F$ and $D_hF_n \to D_hF$ converge locally uniformly in $\mathbb{G}$. Then we consider the corresponding level set of F_n that is also a $\mathbb{KH}$-graph with a smooth (by classical implicit function theorem) graph-map ϕ_n. It turns out that ϕ_n is a desired smooth approximation of ϕ. We can refer to [ASV06; CM06b; MV12; CMPS14] to see the implementation of this strategy.

4.4. Computations examples

In this section we perform explicit calculations of the base projector Π for some splittings in our favourite Carnot groups.

4.4.1. Graphs in Engel group

For the left and right translations in Engel group (see Section 2.2.2) we obtain the following:

$$dL_{(v,x,y,z)}(v', x', y', z') = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{x}{2} & \frac{v}{2} & 1 & 0 \\ -\frac{6y+v(x'+x)}{12} & \frac{v(v-v')}{12} & \frac{v}{2} & 1 \end{pmatrix}.$$

$$dR_{(v,x,y,z)}(v',x',y',z') = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{x}{2} & -\frac{v}{2} & 1 & 0 \\ \frac{6y - v(x'+x)}{12} & -\frac{v(v'-v)}{12} & -\frac{v}{2} & 1 \end{pmatrix}.$$

V-**graph.** Consider the splitting $\mathbb{E} = \mathbb{K} \ltimes \mathbb{H}$ where

$$\mathbb{K} = \{v = 0\}, \quad \mathbb{H} = \{\exp(tV) \mid t \in \mathbb{R}\}.$$

Take

$$a = (v,0,0,0) \cdot (0,x,y,z) = \big(v, x, y - \frac{vx}{2}, z - \frac{vy}{2} + \frac{v^2 x}{12}\big).$$

Then

$$\Pi_a \circ dL_a(e) = dL_{(0,x,y,z)}(e) \circ dL_{(v,0,0,0)}(-v,0,0,0) \circ dR_{(-v,0,0,0)}(e) \circ \pi_{\mathfrak{k}} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{x}{2} & 0 & 1 & 0 \\ -\frac{y}{2} & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{v}{2} & 1 & 0 \\ 0 & \frac{v^2}{6} & \frac{v}{2} & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{v}{2} & 1 & 0 \\ 0 & \frac{v^2}{12} & \frac{v}{2} & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & v & 1 & 0 \\ 0 & \frac{v^2}{2} & v & 1 \end{pmatrix}.$$

(Here, the symbol $\times$ stands for the matrix product.) This gives us

$$\Pi_a\langle X\rangle = \partial_x + v\partial_y + \frac{v^2}{2}\partial_z, \quad \Pi_a\langle Y\rangle = \partial_y + v\partial_z, \quad \Pi_a\langle Z\rangle = \partial_z. \tag{4.23}$$

X-**graph.** Consider now the splitting $\mathbb{E} = \mathbb{K} \ltimes \mathbb{H}$ where

$$\mathbb{K} = \{x = 0\}, \quad \mathbb{H} = \{\exp(tX) \mid t \in \mathbb{R}\}.$$

Take

$$a = (0,x,0,0) \cdot (v,0,y,z) = \big(v, x, y + \frac{vx}{2}, z + \frac{v^2 x}{12}\big).$$

Then

$$\Pi_a \circ dL_a(e) = dL_{(v,0,y,z)}(e) \circ dL_{(0,x,0,0)}(0,-x,0,0) \circ dR_{(0,-x,0,0)}(e) \circ \pi_{\mathfrak{k}} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{v}{2} & 1 & 0 \\ -\frac{y}{2} & \frac{v^2}{12} & \frac{v}{2} & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{x}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{x}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -x & 0 & 1 & 0 \\ -\frac{y+xv}{2} & 0 & \frac{v}{2} & 1 \end{pmatrix}.$$

This means that

$$\Pi_a\langle V\rangle = \partial_v - x\partial_y - \frac{y+xv}{2}\partial_z, \quad \Pi_a\langle Y\rangle = \partial_y + \frac{v}{2}\partial_z, \quad \Pi_a\langle Z\rangle = \partial_z.$$

4.4.2. Graphs in Heisenberg group

X_1-graph. Consider the splitting $\mathbb{H}^n = \mathbb{K} \ltimes \mathbb{H}$ in Heisenberg group where

$$\mathbb{K} = \{x_1 = 0\}, \quad \mathbb{H} = \{\exp(tX_1) \mid t \in \mathbb{R}\}.$$

The differentials of left and right translations read as follows

$$dL_{(x,y,z)}(x', y', z') = \left(\begin{array}{c|c|c} 1_{n\times n} & 0_{n\times n} & 0_n \\ \hline 0_{n\times n} & 1_{n\times n} & 0_n \\ \hline -2y_1 \ldots -2y_n & 2x_1 \ldots 2x_n & 1 \end{array}\right).$$

$$dR_{(x',y',z')}(x, y, z) = \left(\begin{array}{c|c|c} 1_{n\times n} & 0_{n\times n} & 0_n \\ \hline 0_{n\times n} & 1_{n\times n} & 0_n \\ \hline 2y'_1 \ldots 2y'_n & -2x'_1 \ldots -2x'_n & 1 \end{array}\right).$$

Then for

$$a = (0, x_2, \ldots, x_n, y_1, \ldots, y_n, z) \cdot (x_1, 0, \ldots, 0) = (x_1, x_2, \ldots, x_n, y_1, \ldots, y_n, z + 2x_1 y_1)$$

we derive that

$$\begin{aligned} &\Pi_a\langle Y_1\rangle = \partial_{y_1} - 4x_1\partial_z, \\ &\Pi_a\langle Y\rangle = Y, \text{ if } Y \in \{X_2, \ldots, X_n, Y_2, \ldots, Y_n, Z\}. \end{aligned}$$

4.5. Some counterexamples

For regular hypersurfaces in the Heisenberg groups the condition **2.B** in Theorem 4.3.1 is in fact a consequence of the condition **2.A**, see [ASV06, Th. 1.3], [BS10b, Th. 1.2] (the key argument can be found in [ASV06, Th. 5.8]). Even more is true. As we explained above, since for hypersurfaces the mapping ϕ is scalar, for all $X \in \mathfrak{k} \cap \mathfrak{g}_1$ we can consider a weak (distributional) formulation corresponding to **2.A**

$$\Pi_{\Phi(x,y,z)}\langle X\rangle\phi = w_X.$$

In the Heisenberg groups this weak formulation turns out to be equivalent to **2.A** and in fact characterises regular hypersurfaces, see [BS10a, Th. 1.2].

We think that this relaxation is also valid for regular hypersurfaces in all two-steps Carnot groups and this phenomenon is one of manifestations of the rectifiability for the sets of finite perimeter in this case, see [FSS03b].

In the examples below we are going to show that condition **2.B** cannot be dropped in Theorem 4.3.1, i. e. in general a regular hypersurface cannot be characterized only by it horizontal geometry. Such examples are not unexpected and have already appeared in [FSS03b]. Since they cannot take place in a two-step Carnot group and as a framework we choose the Engel group E^4 of three steps (see also [BL13] for some interesting studies of sets in this framework).

Example 4.5.1. So, let $\phi\colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ be a continuous function, used as a graph-map of a V-graph in E^4, see notations in Section 4.4.1. The base projector acts on the only horizontal vector field $X \in \mathfrak{k} \cap \mathfrak{g}_1$ as

$$\hat{X}(x,y,z) = \Pi_{\Phi(x,y,z)}\langle X\rangle = \partial_x + \phi(x,y,z)\partial_y + \frac{\phi(x,y,z)^2}{2}\partial_z.$$

For simplicity, we will take $\phi = \phi(x,z)$ independent of y.

Let a curve $\gamma(x) = (x, z(x))$ be the projection on the (x,z)-plane of some integral curve $\operatorname{Exp}_\phi(\cdot X)((0,y_0,z_0))$ of $\hat{X}$. By definition, the function $z(x) \in C^1$ satisfies

$$z'(x) = \frac{\phi^2 \circ \gamma(x)}{2}.$$

Since ϕ does not depend on $\mathbf{y}$, condition **2.A** reads

$$(\phi \circ \gamma)'(t) = w(\gamma(t)) \tag{4.24}$$

for some scalar continuous function $w = w(x,z)$. Therefore, $z \in C^2$ and

$$z''(x) = \phi(\gamma(x))w(\gamma(x)) = w(\gamma(x))\big(\phi(0,z_0) + \int_0^x w(\gamma(s))\,ds\big),$$

where we denote $z_0 = z(0)$. Integrating this twice we obtain the integral representation of $z(x)$,

$$z(x) = z_0 + \frac{x}{2}\phi(0,z_0)^2 + \phi(0,z_0)\int_0^x w(\gamma(s))(x-s)\,ds + \frac{1}{2}\int_0^x \Big(\int_0^s w(\gamma(\eta))\,d\eta\Big)^2 ds.$$

Let us take now $w = -1$ and $\phi(0,z_0) = -z_0^{\frac{1}{3}}$ that gives us

$$z(x,z_0) = z_0 + \frac{x}{2}|z_0|^{\frac{2}{3}} + \frac{x^2}{2}z_0^{\frac{1}{3}} + \frac{x^3}{6}.$$

We observe that for any point $(x,z) \in \mathbb{R}^2$ there is a unique point z_0 such that $z(x,z_0) = z$ and, moreover, the point $z_0 = z_0(x,y)$ depends continuously on (x,z). Indeed, if we put $a = z_0^{\frac{1}{3}}$ then we need to solve the equation of degree 3 in a:

$$P_x(a) - z = 0 \quad \text{with} \quad P_x(a) = a^3 + \frac{x}{2}a^2 + \frac{x^2}{2}a + \frac{x^3}{6}.$$

Since the derivative $P_x'(a) = 3a^2 + xa + x^2/2 > 0$, the function P_x is strictly monotone if $|x| + |a| > 0$, and, therefore, it can be continuously inversed. It is also obvious that

$z_0(x,z) \to 0 = z_0(0,0)$ when $(x,z) \to (0,0)$, and, thus, z_0 is continuous on $\mathbb{R}^2$. This mean that the integral trajectories $(x, z(x, z_0))$ are not crossing for different z_0 and form a continuous flow on $\mathbb{R}^2$.

Recalling that $(\phi \circ \gamma)'(t) = w(\gamma(t)) = -1$ we find that the function $\phi(z,x) = -z_0(x,z)^{\frac{1}{3}} - x$ is continuous and, by construction, satisfies Eq. (4.24). However, $\phi \notin \mathrm{hol}^{\frac{1}{3}}$ along the trajectories of $\partial_z = \Pi_{\Phi(x,y,z)}\langle Z\rangle$.

Remark 4.5.2. Imagine that we consider only continuous mappings ϕ that don't depend on y. Then, in both $\mathbb{H}^1$ and E^4 we are dealing with a week equation

$$\partial_x \phi + \partial_z G(\phi) = w,$$

where (up to the constant factor) $G_{\mathbb{H}^1}(h) = h^2$ and $G_{E^4}(h) = h^3$. Note that $G_{\mathbb{H}^1}$ is a convex function (giving raise to the Burger's equation) and G_{E^4} is not. And it is well known that this difference is fundamental in the theory of quasi-linear equations (for the notion of entropy solutions, regularity results and so on). For instance, if a continuous function ϕ is a global weak solution for $G_{\mathbb{H}^1}$ with constant $w \equiv C$, then it implies according to the results in [BS10a], that ϕ describes a boundary of sets of with constant horizontal normal, so that, $\phi(x,z) = Cx$ by rectifiability.

The example below emphasises this difference.

Example 4.5.3. Let us take a function $\phi(x,y,z) = z^{\frac{1}{3}}$. Since $\phi^3 \equiv z \in C^1$, it is clear that ϕ satisfies the equation

$$\Pi_{\Phi(x,y,z)}\langle X\rangle \phi = \partial_x \phi + \frac{1}{2}\partial_y \phi^2 + \frac{1}{6}\partial_z \phi^3 = w, \tag{4.25}$$

in the weak sense with constant $w = 1/6$. However, $\phi \notin \mathrm{hol}^{\frac{1}{3}}$ (we don't even speak about some regularity improvement like in [BS10b, Th. 1.4]). We note also that there are characteristics $\gamma(t)' = \Pi_{\Phi(\gamma(t))}\langle X\rangle$ such that $(\phi \circ \gamma)'(t) \neq w$, for instance $\gamma(t) = (t,0,0)$. In fact, $\gamma(t) = (t,0,t^3)$ is the only one among all the characteristics starting at the origin for which $(\phi \circ \gamma)'(t) = w$.

Example 4.5.4. A C^∞-function on $\Omega = \mathbb{R}^3 \setminus \{(0,-c,0)\}$, $c \in \mathbb{R}$,

$$\phi(x,y,z) = \frac{2z}{y+c}$$

is a in fact a strong solution of Eq. (4.25) with $w = 0$. This example is the coordinate representation of the set with constant horizontal normal from [AKL09, Section 7.4]. This tells us that there is no hope to control the vertical (i. e. along integral lines of $\Pi_{\Phi(x,y,z)}\langle Y\rangle$ and $\Pi_{\Phi(x,y,z)}\langle X\rangle$) behaviour of ϕ by some norm of w.

4.6. Application to the rectifiability problem

Here we show what one can derive about the tangent cones of Lipschitz graphs of higher codimension in the Heisenberg groups using the rectifiability in codimension one and Theorem 4.2.16. As a basic model we will consider the case of a Lipschitz graph of codimension 2 in $\mathbb{H}^2$.

Theorem 4.6.1. *Let $S \subset \mathbb{H}^2$ be a Lipschitz graph w. r. t. the splitting $\mathbb{H}^2 = \mathbb{K} \ltimes \mathbb{H}$ where the horizontal subgroup $\mathbb{H}$ is of dimension two, $\dim \mathbb{H} = 2$. We also assume that $\pi_{\mathbb{K}}(S) = \Omega$ is open in $\mathbb{K}$. For $\mathcal{H}^{2\times 2+2-2}$-every point $p \in S$ the following holds. If $\delta_{1/r_i}(p^{-1} \cdot S) \to W$ locally converges in the Hausdorff distance for some $\{r_i\}_{i\geq 0} \subset \mathbb{R}_+$, $r_i \to 0$, then W is a vertical plane.*

Proof. The proof contains several steps.

Coordinates setup. We can assume without loss of generality that $S \subset \mathbb{H}^2$ is a Lipschitz graph w. r. t. the following decomposition:

$$\begin{aligned}
&\mathbb{K} = \exp(\mathrm{span}(\{X_1, Y_2, Z\})) \text{ with coordinates } \mathbf{y} = (x, y, z),\\
&\mathbb{H} = \exp(\mathrm{span}(\{X_2, Y_1\})) \text{ with coordinates } \phi = (\phi_1, \phi_1),\\
&S = \big\{(x, 0, 0, y, z) \cdot (0, \phi_1(x, y, z), \phi_2(x, y, z), 0, 0) \mid (x, y, z) \in \Omega \subset \mathbb{K}\big\}.
\end{aligned}$$

Indeed, the result we want to prove is local and therefore, by Proposition 4.2.8, it depends only on the factor $\mathbb{H}$. Since there is a transitive symplectic action on horizontal subgroups in $\mathbb{H}^2$, we can choose any of such subgroups. And for $\mathbb{H}$ chosen as above we are also free to fix an arbitrary complementary vertical subgroup $\mathbb{K}$. We made the above choice for the simplicity of calculations.

Imagine that we want to prove the result at some point $p \in S$. Let us suppose (by left-translation) that $p = e$, so that, $\delta_{1/r_i}(S) \to W$. It is clear that W is also a $\mathbb{K}\mathbb{H}$-Lipschitz graph with $\mathrm{Lip}(W) \leq \mathrm{Lip}(S)$ and let $\tilde{\phi} = (\tilde{\phi}_1, \tilde{\phi}_2)$ be the graph-map of W. It is easy to see that being a vertical plane for W is equivalent for the function $\tilde{\phi}$ to be linear w. r. t. to horizontal coordinates:

$$\tilde{\phi}_1(x, y, z) = ax + by, \quad \tilde{\phi}_2(x, y, z) = cx + dy.$$

We recall that $\phi_r(x, y, z) = r^{-1}\phi(rx, ry, r^2 z)$ is the graph-map of $\delta_{1/r}(S)$. We observe that the family of functions $\{\phi_r\}_{r>0}$ is equi-continuous (because $\mathrm{Lip}(\delta_{1/r}(S)) \leq \mathrm{Lip}(S)$) so for any sequence $\{r_i\}_{i\geq 0}$ there is a subsequence $\{r_{i_j}\} \subset \{r_i\}$ such that $\phi_{r_{i_j}}$ converges locally uniformly. Therefore, the local convergence $\delta_{1/r_i}(S) \to W$ in the Hausdorff distance is equivalent to the locally uniform convergence

$$\phi_{r_i} := \delta_{1/r_i} \circ \phi \circ \delta_{r_i} \to \tilde{\phi}, \quad i \to \infty.$$

We know by Lemma 4.2.10 that for a Lipschitz graph the Hausdorff measure $\mathcal{H}^4 \llcorner S$ is locally comparable to $\Phi_\#(\mathcal{L}^3 \llcorner \Omega)$. Thus, a subset $S_0 \subset S$ is $\mathcal{H}^4$-negligible on S if and only if $\pi_{\mathbb{K}}(S_0)$ is $\mathcal{L}^3$-negligible.

Codimension one slicing. In the coordinates we choose, the action of the base projector reads as follows

$$\begin{aligned}
\tilde{X}_1 &:= \Pi_{\Phi(x,y,z)}\langle X_1 \rangle = \partial_x - 4\phi_1(x, y, z)\partial_z,\\
\tilde{Y}_2 &:= \Pi_{\Phi(x,y,z)}\langle Y_2 \rangle = \partial_y + 4\phi_2(x, y, z)\partial_z,
\end{aligned}$$

$$\tilde{Z} := \Pi_{\Phi(x,y,z)}\langle Z\rangle = \partial_z.$$

Theorem 4.2.16 says in particular that ϕ_1 is (locally) Lipschitz along the integral lines of $\tilde{X}_1$ and $\mathrm{Hol}^{1/2}$ along the integral lines of $\tilde{Z}$. Let us consider $y_0 \in \mathbb{R}$ for which the plane $\{y = y_0\} \subset \mathbb{K}$ intersects Ω. For these y_0 we denote $\Omega_{y_0} = \Omega \cap \{y = y_0\}$. We view the restriction $\phi_1 \llcorner \Omega_{y_0}$ as a graph-map for $\mathbb{K}_1\mathbb{H}_1$-graph in the first Heisenberg group $\mathbb{H}^1$ with $\mathbb{H}_1 = \exp(\mathrm{span}\{X_1\}) < \mathbb{H}^1$ and $\mathbb{K}_1 = \{x_1 = 0\} \trianglelefteq \mathbb{H}^1$ (with the appropriate notations in $\mathbb{H}^1$). The vector fields $\tilde{X}_1$ and $\tilde{Z}$ with fixed $y = y_0$ have exactly the same structure as if they were in $\mathbb{H}^1$ with the graph-map ϕ_1. The plane $\{y = y_0\}$ is obviously invariant under the generalized flows of $\tilde{X}_1$ and $\tilde{Z}$, i. e. if some integral line of one of these vectorfields starts on $\{y = y_0\}$ then it will not leave this plane. Thus, using the opposite implication in Theorem 4.2.16 we obtain that $\phi_1 \llcorner \Omega_{y_0}$ describes a locally Lipschitz graph in $\mathbb{H}^1$. Let us now exploit at maximum this fact.

At this point we use a characterization of the codimension one Lipschitz graphs in Heisenberg groups given in [BCC14, Th. 1.1]: there is a function $w^1_{y_0} \in L^\infty_{loc}(\Omega_{y_0})$, such that the equation

$$\tilde{X}_1(\phi_1 \llcorner \{y = y_0\}) = w^1_{y_0}$$

holds in distributional sense, i. e. for any test function $f \in C^1(\Omega_{y_0})$ with compact support $\mathrm{supp}(f) \Subset \Omega_{y_0}$ the following is verified

$$\int_{\Omega_{y_0}} \phi_1 \partial_x f - 2\phi_1^2 \partial_z f = -\int_{\Omega_{y_0}} w^1_{y_0} f. \tag{4.26}$$

One can also use a smooth approximation result [CMPS14, Th. 1.7] to deduce this distributional equation.

The Eq. (4.27) comes with an estimate on $w^1_{y_0}$. Locally the L^∞-norm of $w^1_{y_0}$ can be controlled by the Lipschitz constant of graph $\mathbb{K}_1\mathbb{H}_1$-graph, that in its turn is controlled (through the control in Theorem 4.2.16) by the Lipschitz constant of S. In other words, for any $\Omega'_{y_0} \Subset \Omega_{y_0}$ there is a constant $C = C(\Omega'_{y_0}) > 0$ such that $\|w^1_{y_0} \llcorner \Omega'_{y_0}\|_\infty \leq C\,\mathrm{Lip}(S)$. The constant $C(\Omega'_{y_0})$ stays in fact uniformly bounded (for variable y_0) as long as the distance from $\partial\Omega'_{y_0}$ to $\partial\Omega_{y_0}$ is bounded from below by some fixed $\epsilon > 0$. Therefore, using Fubini's theorem and Eq. (4.27) we can derive that for any $\Omega' \Subset \Omega$ there is a constant C such that

$$\left| \int_{\Omega'} \phi_1 \partial_x f - 2\phi_1^2 \partial_z f \right| \leq C\,\mathrm{Lip}(S) \left| \int_{\Omega'} f \right|. \tag{4.27}$$

for any $f \in C^1(\Omega)$ with $\mathrm{supp}(f) \subset \Omega'$. Therefore, by Riesz' representation theorem, there is a function $w^1 \in L^\infty_{loc}(\Omega)$ such that $\tilde{X}_1\phi_1 = w^1$ holds in distributional sense in Ω.

We consider the following family of sets on $\mathbb{K}$:

$$\mathcal{E} = \{\sigma_{\Phi(\mathbf{y})}(E_e) \mid \mathbf{y} \in \Omega\}, \quad E_e = \{\bar{B}(e, r) \cap \mathbb{K} \mid r \in (0, 1)\}.$$

The Lebesgue measure $\mathcal{L}^3$ on $\mathbb{K}$ is doubling w. r. t. this family of sets (see [Fed69, section2.8]). Indeed, the family $\mathcal{E}$ is merely a projection on $\mathbb{K}$ of the family[1] $\tilde{\mathcal{E}} =$

[1]with an appropriate choice of the homogeneous metric ρ as in Lemma 4.2.10

$\{B(a,r) \mid a \in S, r \in (0,1)\}$ that is clearly doubling w. r. t. $\mathcal{H}^4 \llcorner S$ since, for instance, it is (locally) Ahlfors regular. This means that $\mathcal{L}^3$-almost every point $\mathbf{y} \in \Omega$ is a density point of w^1 w. r. t. $\mathcal{E}$. Recall that being a density point means the following:

$$\fint_{\sigma_{\Phi(\mathbf{y})}(B_r(e)\cap\mathbb{K})} |w^1 - w^1(\mathbf{y})|\, d\mathcal{L}^3 \to 0, \quad r \to 0.$$

Since the definition of $\mathcal{E}$ is left-invariant, let us suppose that $\mathbf{y} = (0,0,0)$ is such a point and $\phi(\mathbf{y}) = 0$. Take any subsequence of $\{r_i\}_{i\geq 0}$, $r_i \to 0+$, such that the limit (w. r. t. the local uniform convergence) $\phi_{r_i} \to \tilde{\phi}$ exists.

Since $\mathbf{y}$ is an interior point of Ω, $\tilde{\phi}$ is a graph-map defined on the whole of $\mathbb{K}$.

Let $f \in C^1(\mathbb{K})$ be a test function with compact support. Then

$$\int_{\mathbb{K}} \tilde{\phi}_1 \partial_x f - 2\tilde{\phi}_1{}^2 \partial_z f = \lim_{i\to\infty} \int_{\mathbb{K}} \phi_{1,r_i} \partial_x f - 2\phi_{1,r_i}^2 \partial_z f$$

because starting from some r_i small enough, $\delta_{r_i}(\mathrm{supp}(f)) \Subset \Omega' \Subset \Omega$ and we can use local uniform convergence under the integral sign. Next, by a change of variables we obtain that

$$\begin{aligned}\int_{\mathbb{K}} \phi_{1,r_i} \partial_x f - 2\phi_{1,r_i}^2 \partial_z f &= r_i^{-4} \int_{\mathbb{K}} \phi_1 \partial_x (f \circ \delta_{1/r_i}) - 2\phi_1^2 \partial_z (f \circ \delta_{1/r_i}) \\ &= r_i^{-4} \int_{\mathbb{K}} w^1 (f \circ \delta_{1/r_i}).\end{aligned}$$

Since we have taken the origin $\mathbf{y} = 0$ as a density point of w^1, a convergence takes place in the last term when $i \to \infty$,

$$\begin{aligned}\left| r_i^{-4} \int_{\mathbb{K}} w^1 (f \circ \delta_{1/r_i}) - w^1(0) \int_{\mathbb{K}} f \right| &\leq r_i^{-4} \left| \int_{\mathbb{K}} \big(w^1 - w^1(0)\big)(f \circ \delta_{1/r_i}) \right| \\ &\leq \|f\|_\infty\, r_i^{-4} \int_{\delta_{r_i}(\mathrm{supp}(f))} \left| w^1 - w^1(0) \right| \\ &\leq \|f\|_\infty\, C^4 \fint_{B_{Cr_i}(e)} \left| w^1 - w^1(0) \right| \to 0,\end{aligned}$$

where C is the supremum of r such that $\delta_r(\mathrm{supp}(f)) \subset B_1(e)$.

Thus, any limit function $\tilde{\phi}_1$ satisfies the distributional equation in Ω

$$\partial_x \tilde{\phi}_1 - 4\tilde{\phi}_1 \partial_z \tilde{\phi}_1 = w^1(0)$$

with a constant right-hand side. Since there is no derivative w. r. t. y in this equation and $\tilde{\phi}_1$ is continuous, this equation holds also when restricted on every slice $\{y = y_0\} \cap \mathbb{K}$. Therefore, for a fixed y_0, a continuous function $\tilde{\phi}_{1,y_0} = \tilde{\phi}_1 \llcorner \{y = y_0\}$ solves the distributional Burgers equation with continuous right hand side. In this situation we can apply the result from [Daf06, Th. 1] (see also its generalization [Big10, Th. 1.4.15] for $w^1(0) \not\equiv 0$). It says that $\tilde{\phi}_{1,y_0}$ is also a solution along characteristics, i. e. for any integral curve $\gamma(t) = (t, z(t))$ such that $z'(t) = -2\tilde{\phi}_{1,y_0}(\gamma)$ we have that $(\tilde{\phi}_{1,y_0} \circ \gamma)'(t) = w^1(0)$. From this it is easy to deduce that $\tilde{\phi}_{1,y_0} = w^1(0)x + C$ where C is some constant.

Indeed, $\tilde{\phi}_{1,y_0}$ is linear along characteristics which, as a consequence, are parabolic

$$\gamma(t) = (t, -2w^1(0)t^2 - 2\tilde{\phi}_{1,y_0}(0, z)t + z), \quad z \in \mathbb{R}.$$

Suppose that there are z_1 and z_2 such that $\tilde{\phi}_{1,y_0}(0, z_1) \neq \tilde{\phi}_{1,y_0}(0, z_1)$. This immediately leads to a contradiction with the continuity of $\tilde{\phi}_{1,y_0}$ because in this case two characteristics

$$\gamma_1(t) = (t, -2w^1(0)t^2 - 2\tilde{\phi}_{1,y_0}(0, z_1)t + z_1),$$
$$\gamma_2(t) = (t, -2w^1(0)t^2 - 2\tilde{\phi}_{1,y_0}(0, z_2)t + z_2),$$

will cross each other at the point

$$t = \frac{z_2 - z_1}{2(\tilde{\phi}_{1,y_0}(0, z_2) - \tilde{\phi}_{1,y_0}(0, z_1))}$$

and the limits of $\tilde{\phi}_{1,y_0}$ at the intersection point along each of them will obviously be different.

Thus, we obtained finally that $\tilde{\phi}_1(x, y, z) = w^1(0)x + C(y)$ for any limit function $\phi_{1,r_i} \to \tilde{\phi}_1$. Observe that the function C here may depend on the choice of the dilation sequence $\{r_i\}$ but $w^1(0)$ does not.

Looking at different directions. The slicing argument that we presented above can be applied in exactly the same way (only with different $\mathbb{H}_1$ and $\mathbb{K}_1$ in $\mathbb{H}^1$) for $\phi_2 \llcorner \{x = const\}$ and $\phi_2 - \phi_1 \llcorner \{x - y = const\}$. This corresponds in fact to the three distributional equations that hold for ϕ in Ω:

$$\begin{aligned} \partial_x \phi_1 - 2\partial_z \phi_1^2 &= w^1 \in L^\infty_{loc}(\Omega), \\ \partial_y \phi_2 + 2\partial_z \phi_2^2 &= w^2 \in L^\infty_{loc}(\Omega), \\ (\partial_x + \partial_y)(\phi_2 - \phi_1) + 2\partial_z(\phi_2 - \phi_1)^2 &= w^3 \in L^\infty_{loc}(\Omega). \end{aligned}$$

Let $\tilde{\Omega} \subset \Omega$ be a full measure set that is an intersection of density point sets of the functions w^i, $i = 1, 2, 3$. By left-translation, let us assume that $\mathbf{y} = 0 \in \tilde{\Omega}$ and let $\{r_i\}_{i \geq 0}$, $r_i \to 0+$, be a sequence such that the limit $\phi_{r_i} \to \tilde{\phi}$ exists. Then the limit function $\tilde{\phi}$ satisfies

$$\begin{aligned} \tilde{\phi}_1(x, y, z) &= w^1(0)x + C_1(y), \\ \tilde{\phi}_2(x, y, z) &= w^2(0)y + C_2(x), \\ (\tilde{\phi}_2 - \tilde{\phi}_1)(x, y, z) &= w^3(0)(x + y) + C_3(x - y), \end{aligned} \tag{4.28}$$

where the functions C_1, C_2, C_3 may depend on $\{r_i\}$. Note that $\tilde{\phi}$ does not depend on z and recall that $\tilde{\phi}$ describes a locally Lipschitz graph. Then, applying again Theorem 4.2.16, we see that these functions C_1, C_2, C_3 are indeed Euclidean Lipschitz (as well as $\tilde{\phi}$) and, therefore, they are differentiable almost everywhere. By taking the derivatives (where it is possible) w. r. t. y in the equality

$$w^2(0)y + C_2(x) - w^1(0)x - C_1(y) = w^3(0)(x+y) + C_3(x-y)$$

we obtain

$$w^2(0) - C_1'(y) = w^3(0) - C_3'(x-y).$$

The left-hand side does not depend on x, so must do the right-hand side. This leads us to the conclusion that all functions C_1, C_2, C_3 are in fact linear. Therefore, any limit of $\delta_{1/r_i}(S)$ is a vertical plane described by a graph-map $\tilde{\phi}$ of the form

$$\begin{aligned} \tilde{\phi}_1(x,y,z) &= w^1(0)x + Ky, \\ \tilde{\phi}_2(x,y,z) &= w^2(0)y + (2w^3(0) + w^1(0) - w^2(0) + K)x. \end{aligned} \tag{4.29}$$

with some constant K that may still depend on $\{r_i\}_{i\geq 0}$. $\square$

Remark 4.6.2. We should mention that the system Eq. (4.28) should be completed by one missing independent equation whose the left-hand side can be one of two following expressions

$$\begin{aligned} \tilde{X}\phi_2 &= \partial_x\phi_2 - 4\phi_1\partial_z\phi_2, \\ \tilde{Y}\phi_1 &= \partial_x\phi_1 + 4\phi_2\partial_z\phi_1. \end{aligned}$$

We could not find an obvious way to give a distributional meaning to this equation.

Remark 4.6.3. Let $\mathbf{y} \in \tilde{\Omega}$ and $\gamma(t) := \mathrm{Exp}_\phi(tY)(\mathbf{y})$ is an arbitrary integral curve of $\tilde{Y}$. Then is easy to check that the constant K in Eq. (4.29) is independent of the choice of $\{r_i\}_{i\geq 0}$ (i. e. the tangent is unique) if and only if the derivative $(\phi_1 \circ \gamma)'(0)$ exists (in which case it must be equal to K).

We know that ϕ_1 is Lipschitz along the integral curves of $\tilde{Y}$. If, for instance, ϕ_2 was Euclidean Lipschitz then the integral curves $\mathrm{Exp}_\phi(\cdot Y)(\mathbf{y})$ would form a Lipschitz continuous flow and using the standard Rademacher theorem we would easily derive that the derivative $(\phi_1 \circ \mathrm{Exp}_\phi(tY)(\mathbf{y}))'(0)$ exists for almost every $\mathbf{y} \in \Omega$.

5. Roughness of vertical curves in Heisenberg group

5.1. Introduction to the problem

This chapter is devoted to the local study of one dimensional level sets in Heisenberg groups. To be more precise, we are going to consider the solution of the equation $\{F = 0\}$ with $F \in C^1_h(\mathbb{H}^n, \mathbb{R}^{2n})$ from the metric point of view. We assume as in the classical implicit function theorem that the horizontal differential is nondegenerate, i. e. $\mathrm{Im}(D_h F(a)) = \mathbb{R}^{2n}$. Thus, the kernel of $\mathrm{Ker}\, D_h F(a) = \mathbb{Z}$ is the one-dimensional centre of $\mathbb{H}^n$ (we refer to this fact when speaking about one dimensional level sets). In Heisenberg groups framework, it is common to call $\mathbb{Z}$ the *vertical axis* (it is where the term "vertical curve" comes from).

This very particular and rather simple case turns out to be very fertile and opens the horizon for further investigations. However, many important questions stay unsolved. Let us mention as an important example the validity of the coarea formula for $F \in C^1_h(\mathbb{H}^n, \mathbb{R}^{2n})$.

Notation 5.1.1. In this chapter we denote $F \in C^1_h(\mathbb{H}^n, \mathbb{R}^{2n})$ normalized to $F(0) = 0$. We assume $D_h F(0)$ being surjective. We recall the notation for the centre $\mathbb{Z} := \{\exp(tZ) \mid t \in \mathbb{R}\}$ of $\mathbb{H}^n$.

Remark 5.1.2. Since the horizontal sub-bundle (complementary to $\mathbb{Z}$) is non-integrable, the kernel $\mathrm{Ker}\, D_h F(0) = \mathbb{Z}$ does not admit a complementary subgroup.

Naive approach to compute $\mathcal{H}^2$ of $F^{-1}(0)$ if F is smooth. In this case, according to the classical implicit function theorem the set $F^{-1}(0)$ is a smooth curve that admits a local representation

$$F^{-1}(0) \cap U = \big\{\Gamma(z) = (\gamma(z), z) \mid z \in [-\delta, \delta]\big\}, \tag{5.1}$$

where γ takes values in $\mathbb{R}^{2n}$. Moreover, the smoothness of γ guaranties that the map $\{z \to \Gamma(z)\}$ is bi-Lipschitz from $\big([-\delta, \delta], |\cdot|^{\frac{1}{2}}\big)$ to $(\mathbb{H}^n, \mathsf{d})$. Note that Γ is never tangent to the horizontal distribution because $DF(a) \circ \pi$ is injective. Hence, we can compute the density by

$$J(z) = \lim_{t \to 0+} t^{-1} \mathsf{d}\big(\Gamma(z), \Gamma(z+t)\big)^2 = 1 + 2 \lim_{t \to 0+} t^{-1} \mathcal{B}(\gamma(z), \gamma(z+t)) = 1 + 2\mathcal{B}(\gamma, \dot{\gamma}).$$

Function J is continuous and strictly positive. By [Fed69, sec. 2.10.10,2.10.11] we know that $\mathcal{H}^2 \llcorner \Gamma = \Gamma_\# \big(J(\cdot)\mathcal{H}^2_{1/2}\big)$, where $\mathcal{H}^2_{1/2}$ is the Hausdorff measure on $[-\delta, \delta]$ built w. r. t.

$|\cdot|^{\frac{1}{2}}$. Note that $\mathcal{H}^2_{1/2} = \mathcal{L}^1$, and, thus,

$$\mathcal{H}^2(\Gamma) = \int_{-\delta}^{\delta} 1 + 2\mathcal{B}(\gamma, \dot{\gamma})\, dz = \int_{\Gamma} \omega,$$

where ω is the contact form. In general, a smooth simple curve Γ that is never horizontal, up to left translation, admits a local representation of the form Eq. (5.1). Therefore, we showed that for such a curve Γ the Hausdorff measure $\mathcal{H}^2$ is given by the integral of the contact form. Of course, the orientation of Γ is chosen in order to make ω positive on Γ. Further, we will generalize this formula. By the way, the geometric interpretation of measure $\mathcal{H}^2(\Gamma)$ is quite remarkable. For instance, in $\mathbb{H}^1$ it is equal to the difference between the increment of z-coordinate along Γ and four times the algebraic area swept by the projection $\pi(\Gamma)$ in the horizontal plane (see Figure 5.1).

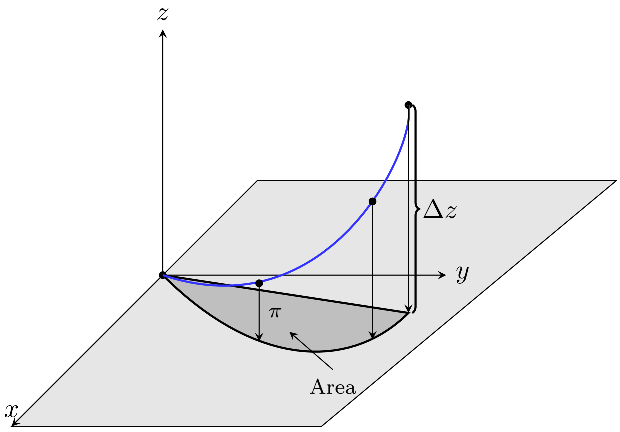

Figure 5.1.: Geometric interpretation of $\mathcal{H}^2(\Gamma)$ in $\mathbb{H}^1$

5.2. Flatness condition

In this section we show that level sets $F^{-1}(0)$ are exactly characterized by the property of being a *Reifenberg flat* w. r. t. the vertical axis $\mathbb{Z}$ with vanishing constant. In particular, the vertical axis plays the role of an approximation to $F^{-1}(0)$ at any point and at every scale. To more precise let us introduce the following definitions and notations.

Notation 5.2.1. For $E \subset \mathbb{H}^n$, $a \in \mathbb{H}^n$ and $r > 0$ we put

- $E_{a,r} := \bar{B}(a, r) \cap E$,
- $\mathbb{Z}_a := L_a(\mathbb{Z}) = \{\exp(Zt)(a) \mid t \in \mathbb{R}\}$,
- $\mathbb{Z}_{a,r} := \bar{B}(a, r) \cap \mathbb{Z}_a$.

Remark 5.2.2. $\mathsf{d}(b, \mathbb{Z}_a) = \mathsf{d}(a, \mathbb{Z}_b) = \|\pi(a^{-1} \cdot b)\| = \|\pi(b) - \pi(a)\|$.

Definition 5.2.3. A closed set $E \subset \mathbb{H}^n$ is called *ε-Reifenberg flat* w. r. t. $\mathbb{Z}$ in $U \subset \mathbb{H}^n$ (starting from the scale $r_0 > 0$) if

$$\operatorname{dist}_{\mathsf{d}}(E_{a,r}, \mathbb{Z}_{a,r}) \leq \varepsilon r \quad \text{for all } a \in E \cap U \text{ and } 0 < r \leq r_0. \tag{5.2}$$

Definition 5.2.4. A closed set $E \subset \mathbb{H}^n$ is called *Reifenberg flat* w. r. t. $\mathbb{Z}$ with *vanishing constant* in $U \subset \mathbb{H}^n$ if there is a $\varepsilon(r) \xrightarrow[r \searrow 0]{} 0$ such that

$$\operatorname{dist}_{\mathsf{d}}(E_{a,r}, \mathbb{Z}_{a,r}) \leq \varepsilon(r) r \quad \text{for all } a \in E \cap U.$$

Proposition 5.2.5. *There is a compact neighbourhood U of 0 such that*

$$\|\pi(a^{-1} \cdot b)\| = o(|z(a^{-1} \cdot b)|^{\frac{1}{2}}), \tag{5.3}$$

that holds uniformly when $a, b \in F^{-1}(0) \cap U$ and $b \to a$. Conversely, if on any compact part of a closed set $E \subset \mathbb{H}^n$ the estimate Eq. (5.3) holds uniformly, then we can find a function $F \in C_h^1(\mathbb{H}^n, \mathbb{R}^{2n})$ such that $E \subset F^{-1}(0)$ and $D_h F(a)$ is surjective for all $a \in E$.

Notation. We are going to call the relation Eq. (5.3) *Whitney's condition.*

Notation 5.2.6. For $F \in C_h^1(\mathbb{H}^n, \mathbb{R}^{2n})$ we introduce some useful quantities characterizing the behaviour of $D_h F$ on $U \subset \mathbb{H}^n$:

$$\begin{aligned}
\omega_F(U) &= \sup\{\|D_h F(a)\langle b\rangle - D_h F(a')\langle b\rangle\| \mid a, a' \in U, \quad b \in \mathbb{H}^n, \quad \|\pi(b)\| = 1\},\\
n_F(U) &= \inf\{\|D_h F(a)\langle b\rangle\| \mid a \in U, \quad b \in \mathbb{H}^n, \quad \|\pi(b)\| = 1\},\\
N_F(U) &= \sup\{\|D_h F(a)\langle b\rangle\| \mid a \in U, \quad b \in \mathbb{H}^n, \quad \|\pi(b)\| = 1\}.
\end{aligned}$$

Proof. As $D_h F$ is surjective at 0 and continuous, we can take a compact neighbourhood $U \subset \mathbb{H}^n$ of 0 such that $D_h F(a)$ is surjective for all $a \in U$. Therefore, for $a \in U$ the value $\|D_h F(a)\langle b\rangle\|$ is equivalent to $\|\pi(b)\|$:

$$n_F(\{a\})\|\pi(b)\| \leq \|D_h F(a)\langle b\rangle\| \leq N_F(\{a\})\|\pi(b)\|.$$

Since $F(a) = F(b)$, Lagrange's Theorem 2.3.4 reads

$$\|D_h F(a)\langle a^{-1} \cdot b\rangle\| \leq C\mathsf{d}(a, b)\omega_F(B(a, C\mathsf{d}(a, b))).$$

Thus,

$$\begin{aligned}
\|\pi(a^{-1} \cdot b)\| &\leq n_F(\{a\})^{-1}\|D_h F(a)\langle a^{-1} \cdot b\rangle\|\\
&\leq C n_F(\{a\})^{-1}\mathsf{d}(a, b)\omega_F(B(a, C\mathsf{d}(a, b))).
\end{aligned} \tag{5.4}$$

Observe that $\omega_F(B(a, C\mathsf{d}(a, b))) \to 0$ when $b \to a$ uniformly for $a \in U$. So, by recalling the definition of d we show Eq. (5.3).

To proof the inverse property, it is enough to put $D_h F(a)\langle b\rangle = \pi(b)$ and $F(a) = 0$ for all $a \in E$ and to apply Whitney's extension Theorem 2.3.6. $\square$

Remark 5.2.7. Thus, according to Whitney's condition, $\mathsf{d}(a, b) = |z(a^{-1} \cdot b)|^{\frac{1}{2}}$ for any $a, b \in F^{-1}(0) \cap U$ close enough.

Remark 5.2.8. We see from Whitney's condition that the concrete values of D_hF on $F^{-1}(0)$ gives no information about $F^{-1}(0)$ (provided, of course, that D_hF is surjective). For instance, D_hF being very regular (like a constant map) along $F^{-1}(0)$ says nothing about the regularity of $F^{-1}(0)$.

Proposition 5.2.9. *Put $h_{a,r} := B(a,r) \cap \exp(H\mathbb{H}^n)(a)$. Let U be like in Proposition 5.2.5. There are constants $r, K > 0$ (depending on F and U) such that for any $a \in F^{-1}(0) \cap U$ and $b \in h_{a,r} \cap U$,*

$$\|F(b)\| \geq K\|\pi(a^{-1} \cdot b)\|.$$

In particular, $h_{a,r} \cap F^{-1}(0) \cap U = \{a\}$ for all $a \in F^{-1}(0) \cap U$.

Proof. Take $a \in F^{-1}(0) \cap U$ and $h_{a,1} \ni b \to a$. From Lagrange theorem,

$$\|F(b) - D_hF(a)\langle a^{-1} \cdot b\rangle\| \leq C\mathsf{d}(a,b)\omega_F(B(a, C\mathsf{d}(a,b))).$$

By the invertibility of $D_hF(a)$ on horizontal plane,

$$\|D_hF(a)\langle a^{-1} \cdot b\rangle\| \geq n_F(U)\|\pi(a^{-1} \cdot b)\|$$

holds for all $a \in U$ and all $b \in h_{a,1}$. One can find $r > 0$ small enough such that for $a \in F^{-1}(0) \cap U$

$$2C\omega_F(B(a, Cr)) \leq n_F(U).$$

Since $a^{-1} \cdot b$ is horizontal (i. e. $z(a^{-1} \cdot b) = 0$), $\mathsf{d}(a,b) = \|\pi(a^{-1} \cdot b)\|$, and the result follows from the triangle inequality with $K = n_F(U)/2$. $\square$

The following Lemma says that the projection of $F^{-1}(0)$ on the vertical axis is surjective.

Lemma 5.2.10. *There are $t > 0$ et $\delta > 0$ such that for any $z \in [-\delta, \delta]$ one can find a point $\gamma_z \in \mathbb{R}^{2n}$ (not necessary unique, see Example A.4.1) with $\|\gamma_z\| < t$ such that $\big(\gamma_z, z\big)$ lies on $F^{-1}(0)$.*

Proof. Denote by $F_z(x,y) = F(x,y,z)$ the restriction of F on planes $z = const$. For any z the map $F_z \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is continuous. Moreover, F_z is differentiable (in classical sense) at $0 \in \mathbb{R}^{2n}$ and $dF_z(0) = D_hF(0,z)$ is invertible.

Take a radius $t > 0$ from Proposition 5.2.9 assuring that for any b in closed disc $D_t = \bar{h}_{0,t} \Subset \mathbb{R}^{2n}$ the estimate $\|F_0(b)\| \geq K\|b\|$, $K > 0$, holds. Therefore, the image of sphere ∂D_t by F_0 does meet the origin. By the continuity of F, there is $\delta > 0$ small enough such that $0 \notin F_z(\partial D_t)$ for all $z \in I = [-\delta, \delta]$. It is clear that any $F_z \in C^0\big(D_t, \mathbb{R}^{2n}\big)$ is homotopic to $F_0 \in C^0\big(D_t, \mathbb{R}^{2n}\big)$ through F. We recall the the topological degree (see [Llo78]) $\deg\big(F_z, D_t, 0\big)$ is homotopy invariant for all $z \in I$, because $0 \notin F_z\big(\partial D_t\big)$.

Let's check that the degree $\deg\big(F_0, D_t, 0\big)$ belongs to $\{-1; 1\}$. Observe that $F_0^{-1}(0) \cap D_t = \{0\}$. Furthermore, F_0 is homotopic (by action of dilations δ_r) to its differential at 0, and thus,

$$\deg\big(F_0, D_t, 0\big) = \deg\big(dF_0(0), \mathbb{R}^{2n}, 0\big) = \deg\big(D_hF(0) \circ \pi, \mathbb{R}^{2n}, 0\big)$$

$$= \operatorname{sign}(\det(\{X_i F(0), Y_i F(0) \mid i = 0, \dots, n\})) \in \{1, -1\}.$$

As for $z \in I$ the degree $\deg\big(F_z, D_t, 0\big) = \deg\big(F_0, D_t, 0\big)$ is different from 0, for every $z \in I$ we can find $\gamma_z \in D_t$ such that $0 = F_z(\gamma_z) = F(\gamma_z, z)$. $\square$

Remark 5.2.11. Let us be more precise about the dependence of t and δ on F. We see that t can be chosen in such a way that

$$\omega_F(h_{0,t}) n_F(\{0\}) \lesssim 1.$$

For δ, we have (through Lip-continuity of F) that

$$\delta^{\frac{1}{2}} \gtrsim t n_F(\{0\}) N_F(B(0,t))^{-1}.$$

Proposition 5.2.12. *There exists a neighbourhood $\tilde{U}$ of $0 \in \mathbb{H}^n$ such that $E = F^{-1}(0)$ satisfies*

$$\operatorname{dist}_{\mathsf{d}}\big(E_{a,r}, \mathbb{Z}_{a,r}\big) \le \varepsilon(r) r \text{ for all } a \in E \cap \tilde{U}, \tag{5.5}$$

with $\varepsilon(r) \to 0$ when $r \to 0$.

Proof. First, consider $0 \in F^{-1}(0)$. If $a \in F^{-1}(0) \cap B(0,r)$, then $\mathsf{d}(a, \mathbb{Z}_{0,r}) = \|\pi(a)\| = o(|z(a)|^{\frac{1}{2}}) = o(r)$, for $r \to 0$ according to Eq. (5.3). If $a = (0,0,z) \in \mathbb{Z}_{0,r}$, then for $r \le t$ small enough thanks to Lemma 5.2.10 one can find $\tilde{a} = (\gamma_z, z) \in F^{-1}(0)$. Again by Eq. (5.3), $\mathsf{d}(a, \tilde{a}) = \|\gamma_z\| = o(r)$ and, thus, $\operatorname{dist}_{\mathsf{d}}\big(E_{0,r}, \mathbb{Z}_{0,r}\big) = o(r)$.

Imitating the proof of Lemma 5.2.10 we show that the continuity of $D_h F$ allows us to choose a compact neighbourhood $\tilde{U}$ of 0 that satisfies the following. Lemma 5.2.10 can be applied to shifted function $F \circ L_{a^{-1}}$ with some $t > 0$ and $\delta > 0$ independent of $a \in F^{-1}(0) \cap \tilde{U}$. Thus, we see that the argument above can be applied (by left translation) to any $a \in F^{-1}(0) \cap \tilde{U}$, that gives $\operatorname{dist}_H\big(E_{A,r}, \mathbb{Z}_{A,r}\big) = o(r)$, where "small-o" is uniform, exactly as in Eq. (5.3). $\square$

Proposition 5.2.13. *Assume that a closed set $E \subset \mathbb{H}^n$ satisfies*

$$\mathsf{d}(b, \mathbb{Z}_{a,r}) \le \varepsilon(r) r \text{ for all } a \in E \cap U \text{ and } b \in E_{a,r},$$

where $\varepsilon(r) \xrightarrow[r \searrow 0]{} 0$ and U is compact. Then there exists a function $F \in C_h^1(\mathbb{H}^n, \mathbb{R}^{2n})$ such that $F(b) = 0$ and $D_h F(b) \circ \pi = \mathrm{Id}_{\mathbb{R}^{2n}}$ for any $b \in E \cap U$.

Proof. As usual, we are going to check the Whitney's condition Eq. (5.3) for $E \cap U$ (see Proposition 5.2.5). We take two points $a, b \in E \cap U$ close enough in such way that $\varepsilon(r) \le 1$ for $r := \mathsf{d}(a,b)$. It is straightforward that $\|\pi(a) - \pi(b)\| = \mathsf{d}(b, \mathbb{Z}_{a,r}) \le \varepsilon(r) r$, and hence, $r = \mathsf{d}(a,b) = |z(a^{-1} \cdot b)|^{\frac{1}{2}}$, and we are done. $\square$

Finally, we obtain the following (local) purely metric characterization of level sets.

Lemma 5.2.14. *In a neighbourhood of the origin, vertical curves of the form $F^{-1}(0)$ are exactly characterized as Reifenberg flat set w. r. t. $\mathbb{Z}$ with vanishing constant.*

Proof. Combine Propositions 5.2.12 and 5.2.13. $\square$

5.3. Reifenberg parametrization

Using Reifenberg flatness we're going to show that locally $F^{-1}(0)$ is a simple curve (i. e. homeomorphic to an interval). We construct a (local) parametrization of $F^{-1}(0)$ by an iterative (from big scales to small ones) dyadic procedure that is quite easy because $\mathbb{Z}$ is of topological dimension 1. This parametrization is not canonical but still enjoys some good metric properties such as being bi-Hölder.

Definition 5.3.1. *The vertical cone* with vertex $a \in \mathbb{H}^n$ and aperture $\varepsilon \in (0,1)$ and of radius $r > 0$ is the compact set

$$\mathcal{C}_{r,\varepsilon}(a) := \{b \in \mathbb{H}^n \mid \mathsf{d}(b, \mathbb{Z}_a) \leq \varepsilon \mathsf{d}(a,b)\} \cap \bar{B}(a,r).$$

The vertical cone is naturally split into $\mathcal{C}_{r,\varepsilon}(a) = \mathcal{C}^+_{r,\varepsilon}(a) \cup \mathcal{C}^-_{r,\varepsilon}(a)$, where

$$\mathcal{C}^{\pm}_{r,\varepsilon}(a) = \mathcal{C}_{r,\varepsilon}(a) \cap \{b \in \mathbb{H}^n \mid z(a^{-1} \cdot b) \gtreqless 0\}.$$

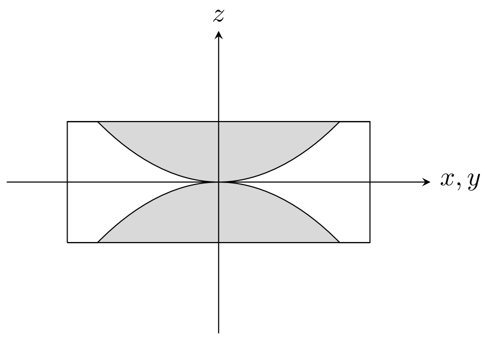

Figure 5.2.: Vertical cone $\mathcal{C}_{r,\varepsilon}(0)$ (grey zone)

Remark 5.3.2. If E is ε-Reifenberg flat w. r. t. $\mathbb{Z}$ in U, then $a \in \mathcal{C}_{r,\varepsilon}(b)$ for all $a, b \in E \cap U$ such that $r \leq \mathsf{d}(a,b) \leq r_0$. Indeed, it suffices to check it for $r = \mathsf{d}(a,b)$ for which we have $\mathsf{d}(a, \mathbb{Z}_{b,r}) \leq \varepsilon r = \varepsilon \mathsf{d}(a,b)$.

Remark 5.3.3. From the triangle inequality on $\mathbb{R}$ we derive the "squared" triangle inequality

$$\mathsf{d}(o,a)^2 \leq \mathsf{d}(o,b)^2 + \mathsf{d}(a,b)^2. \tag{5.6}$$

that holds for $o, a, b \in \mathbb{H}^n$ with $o^{-1} \cdot a \in \mathbb{Z}$.

Proposition 5.3.4. *Take $o, a, b \in \mathbb{H}^n$ with $a \in \mathcal{C}^+_{r,\varepsilon}(b)$, $a \in \mathcal{C}^+_{r,\varepsilon}(o)$ and $b \in \mathcal{C}^-_{r,\varepsilon}(o)$. Then*

$$\frac{\varepsilon^2}{1+\varepsilon^2}\mathsf{d}(a,b)^2 \leq \mathsf{d}(o,a)^2 + \mathsf{d}(o,b)^2 - \mathsf{d}(a,b)^2 \leq \frac{\varepsilon^2}{1-\varepsilon^2}\mathsf{d}(a,b)^2. \tag{5.7}$$

Proof. Since $\varepsilon < 1$, for max-type distance d we have $\mathsf{d}(a,b)^2 = z(b^{-1} \cdot a)$, $\mathsf{d}(o,b)^2 = z(b^{-1} \cdot o)$ and $\mathsf{d}(o,a)^2 = z(o^{-1} \cdot a)$. We can estimate the difference of z-components by the norm of a homogeneous polynomial of degree 2 containing components of $o^{-1} \cdot b$ and $o^{-1} \cdot a$ that are complementary to z, concretely:

$$
\begin{aligned}
&|\mathsf{d}(o,a)^2 + \mathsf{d}(o,b)^2 - \mathsf{d}(a,b)^2| = |z(b^{-1} \cdot a) - z(b^{-1} \cdot o) - z(o^{-1} \cdot a)| \\
&= 2|\mathcal{B}(\pi(b^{-1} \cdot o), \pi(o^{-1} \cdot a))| \leq 2\|\pi(b^{-1} \cdot o)\|\|\pi(b^{-1} \cdot o)\| \\
&\qquad \leq 2\varepsilon^2 \mathsf{d}(o,a)\mathsf{d}(o,b) \leq \varepsilon^2(\mathsf{d}(o,a)^2 + \mathsf{d}(o,b)^2),
\end{aligned}
\tag{5.8}
$$

and Eq. (5.7) follows immediately. □

Theorem 5.3.5 (of parametrization of vertically Reifenberg flat sets)**.** *Let $E \Subset \mathbb{H}^n$ be ε-Reifenberg flat w. r. t. $\mathbb{Z}$ in $\bar{B}(0,R)$ starting from the scale $r_0 > 0$ with $0 < \varepsilon \leq \varepsilon_0$ small enough. Then, locally the set $E \cap B(0,R)$ is a simple curve admitting a bi-Hölder parametrization.*

Proof. Take some point $a \in E \cap B(0,R)$. We put $r = \min\{r_0, \mathsf{d}(a, \partial B(0,R))\}/2$. According to Reifenberg condition Eq. (5.2), there is a point $b \in \mathcal{C}_{r,\varepsilon}(a) \cap E$ such that $\mathsf{d}\big(b, \exp(r^2 Z)(a)\big) \leq \varepsilon r$. In fact, $b \in \mathcal{C}^+_{r,\varepsilon}(a)$ because $B(\exp(r^2 Z)(a), \varepsilon r) \cap \mathcal{C}^-_{r,\varepsilon}(a) = \varnothing$ when $\varepsilon < 1$.

Put now $r_1 = \mathsf{d}(a,b)/\sqrt{2}$. We can find a point $o \in E \cap \mathcal{C}^+_{r_1,\varepsilon}(a)$ "in the middle" between a and b such that $\mathsf{d}\big(o, \exp(r_1^2 Z)(a)\big) \leq \varepsilon \mathsf{d}(a,o)$. This implies

- $|z(a^{-1} \cdot o)| \leq r_1^2$,
- $\max\{\|\pi(a^{-1} \cdot o)\|, |z(a^{-1} \cdot o) - r_1^2|^{\frac{1}{2}}\} = \mathsf{d}\big(o, \exp(r_1^2 Z)(a)\big) \leq \varepsilon d(a,o)$.

By Eq. (5.6) we get

$$
r_1^2 = \mathsf{d}(a, \exp(r_1^2 Z)(a))^2 \leq \mathsf{d}(a,o)^2 + \mathsf{d}(o, \exp(r_1^2 Z)(a))^2 \leq \mathsf{d}(a,o)^2(1+\varepsilon^2),
$$

so $\mathsf{d}(a,o)^2 \geq r_1^2/(1+\varepsilon^2)$.

It is easy to see that $o \in \mathcal{C}^+_{r,\varepsilon}(a) \cap \mathcal{C}^-_{r,\varepsilon}(b)$. So we can use Eq. (5.7) to estimate $\mathsf{d}(o,b)$:

$$
\begin{aligned}
\mathsf{d}(o,b)^2 &\leq \mathsf{d}(a,b)^2 - \mathsf{d}(o,a)^2 + \frac{\varepsilon^2}{1-\varepsilon^2}\mathsf{d}(a,b)^2 \leq (1 + \frac{2\varepsilon^2}{1-\varepsilon^2})r_1^2, \\
\mathsf{d}(o,b)^2 &\geq \mathsf{d}(a,b)^2 - \mathsf{d}(o,a)^2 - \frac{\varepsilon^2}{1+\varepsilon^2}\mathsf{d}(a,b)^2 \leq (1 + \frac{2\varepsilon^2}{1+\varepsilon^2})r_1^2.
\end{aligned}
$$

By summing up our estimates we obtain that

$$
\begin{aligned}
\max\{\mathsf{d}(o,a), \mathsf{d}(o,b)\} &\leq c_+(\varepsilon)\mathsf{d}(a,b), \\
\min\{\mathsf{d}(o,a), \mathsf{d}(o,b)\} &\geq c_-(\varepsilon)\mathsf{d}(a,b),
\end{aligned}
\tag{5.9}
$$

where $c_+(\varepsilon) = (1 + C\varepsilon^2)/\sqrt{2}$ and $c_-(\varepsilon) = (1 - C\varepsilon^2)/\sqrt{2}$ with some universal constant $C > 0$. We assume that ε_0 is small enough in such a way that $c_+(\varepsilon_0) < 1$. In this proof we are not seeking for an optimal value of ε_0.

Now we define recursively a sequence of maps

$$
\Gamma_l \colon T_l \to E, \quad T_l := \{k2^{-l} \mid k = 0, \ldots, 2^l\} \subset [0,1], \quad l \geq 0.
$$

To start with, we set $\Gamma_0(0) = a$ and $\Gamma_0(1) = b$. Then we construct Γ_{l+1} as an extension of Γ_l by mapping a middle point of new dyadic generation to a corresponding "middle" point on E, i. e. $\Gamma_1(\frac{1}{2}) = o$, $\Gamma_2(\frac{1}{4})$ will be a "middle" point between a and o, and so on. Each "middle" point is defined in the same way, i. e. as we've done for o and a, b. The iterated estimates Eq. (5.9) give that

$$\begin{aligned} \max_{k=0,\dots,2^l-1} \mathsf{d}\Big(\Gamma_l\big(\frac{k}{2^l}\big), \Gamma_l\big(\frac{k+1}{2^l}\big)\Big) \le c_+(\varepsilon)^l \mathsf{d}(a,b), \\ \min_{k=0,\dots,2^l-1} \mathsf{d}\Big(\Gamma_l\big(\frac{k}{2^l}\big), \Gamma_l\big(\frac{k+1}{2^l}\big)\Big) \ge c_-(\varepsilon)^l \mathsf{d}(a,b). \end{aligned} \tag{5.10}$$

Merely by equi-continuity of $\{\Gamma_l\}_{l\ge 0}$, there is unique map $\Gamma\colon [0,1] \to E$ that extends Γ_l for all l. We are going to show that the distortion rates Eq. (5.10) together with flatness of Γ assure that Γ satisfies bi-Hölder condition:

$$C_1|t-s|^\beta \le \mathsf{d}(\Gamma(t), \Gamma(s)) \le C_2|t-s|^\alpha \tag{5.11}$$

with the exponents

$$\begin{aligned} \alpha = -\ln(c_+(\varepsilon))/\ln(2) \approx \frac{1}{2} - C\varepsilon^2, \\ \beta = -\ln(c_-(\varepsilon))/\ln(2) \approx \frac{1}{2} + C\varepsilon^2. \end{aligned}$$

In particular, Γ will be simple (injective) curve.

To get the direct Hölder inequality, one can use a classical argument, see for instance [LV07, Lemma 2.]. Let us prove the inverse one. Take $0 \le s \le t \le 1$ with $2^{-l+1} > |t-s| \ge 2^{-l}$ for some $l > 0$. We can always find $s_1, t_1 \in T_{l+2}$ such that $s \le s_1 \le t_1 \le t$ (see Figure 5.3) Note that if $s', t' \in [0,1]$ with $s' \le t'$ then by construc-

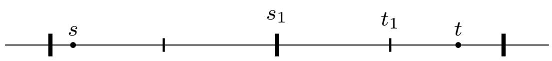

Figure 5.3.: $4|t_1 - s_1| \ge |t-s|$

tion $z(\Gamma(s')^{-1} \cdot \Gamma(t')) \ge 0$. So, by appling Eq. (5.7) we get

$$\begin{aligned} \mathsf{d}(\Gamma(s), \Gamma(t))^2 \ge (1-\varepsilon^2)(\mathsf{d}(\Gamma(s), \Gamma(s'))^2 + \mathsf{d}(\Gamma(s'), \Gamma(t))^2), \\ \mathsf{d}(\Gamma(s'), \Gamma(t))^2 \ge (1-\varepsilon^2)(\mathsf{d}(\Gamma(s'), \Gamma(t'))^2 + \mathsf{d}(\Gamma(t'), \Gamma(t))^2). \end{aligned}$$

Therefore,

$$\mathsf{d}(\Gamma(s), \Gamma(t))^2 \ge (1-\varepsilon^2)^2 \mathsf{d}(\Gamma(s'), \Gamma(t'))^2 \ge (1-\varepsilon^2)^2 c_-(\varepsilon)^{l+2} \mathsf{d}(a,b)^2 \ge C_1 2^{-\beta(l-1)},$$

where $C_1 = \mathsf{d}(a,b)^2(1-\varepsilon^2)^2 c_-(\varepsilon)^3 > 0$.

Let us show that $\Gamma([0,1]) = E \cap U$, where $U = \mathcal{C}^+_{r_0,\varepsilon}(o) \cap \mathcal{C}^-_{r_0,\varepsilon}(a)$. The main observation here is that the "middle" point o we've taken belongs to $E \cap U$ and thanks to flatness condition we get the splitting $E \cap U = (E \cap U_0) \cup (E \cap U_1)$, where

$$U_0 = \mathcal{C}^+_{r_0,\varepsilon}(a) \cap \mathcal{C}^-_{r_0,\varepsilon}(o), \quad U_1 = \mathcal{C}^+_{r_0,\varepsilon}(o) \cap \mathcal{C}^-_{r_0,\varepsilon}(b), \quad \text{so that } U_0 \cap U_1 = \{o\}.$$

So, by iterating this argument we get for any $l \geq 0$

$$E \cap U = E \cap \bigcup_{k=0}^{2^l-1} U_{l,k}, \quad U_{l,k} := \mathcal{C}^+_{r_0,\varepsilon}\big(\Gamma_l\big(\frac{k}{2^l}\big)\big) \cap \mathcal{C}^-_{r_0,\varepsilon}\big(\Gamma_l\big(\frac{k+1}{2^l}\big)\big).$$

Thus, $\Gamma_l(T_l) \xrightarrow[l\to\infty]{} E \cap U$ in terms of the Hausdorff distance because

$$\operatorname{dist}_{\mathsf{d}}(E \cap U, \Gamma_l(T_l)) \leq \max_k \operatorname{diam} U_{l,k} \lesssim \max_k \mathsf{d}(\Gamma_l(\frac{k}{2^l}), \Gamma_l(\frac{k+1}{2^l})) \leq c_+(\varepsilon)^l \mathsf{d}(a,b).$$

We should apply the same arguments to show the existence of a point $\tilde{a} \in E \cap \mathcal{C}^-_{r,\varepsilon}(o)$, $\mathsf{d}\big(\tilde{a}, \exp(-r^2 Z)(o)\big) \leq \varepsilon \mathsf{d}(o, \tilde{a})$, such that $E \cap \mathcal{C}^-_{r_0,\varepsilon}(o) \cap \mathcal{C}^+_{r_0,\varepsilon}(\tilde{a})$ is a simple curve as well. Since $E \cap \bar{B}(o,r) = E \cap \mathcal{C}_{r,\varepsilon}(o)$ for $0 < r \leq r_0$, we've finished the proof. $\square$

Remark 5.3.6. If we are not too much interested in quantitative aspect we can obtain the topological result in Theorem 5.3.5 as follows. We derive first that compact set E is locally connected because the "middle" points can form a ϵ-net on E for any $\epsilon > 0$. Next, we observe that for a couple of points $a, b \in E$ close enough to each other, the linear order $a \leq b \iff z(a^{-1} \cdot b) \geq 0$ is well defined. Thus, the connected set $E \cap U$ endowed with a linear order that respects topology is homomorphic to an interval. We are going to realize this strategy in Theorem 3.4.5.

Notion of vertical curve

For the level sets we can now deduce from Lemma 5.2.14 and Theorem 5.3.5 the following theorem.

Theorem 5.3.7. *Let $F \in C^1_h(\mathbb{H}^n, \mathbb{R}^{2n})$, $F(0) = 0$, be such that the differential of $D_h F(0)$ is surjective. Then there is a neighbourhood U of $0 \in \mathbb{H}^n$ such that $U \cap F^{-1}(0)$ is a simple curve.*

This result give raise to the notion of *vertical curve*.

Notation 5.3.8. We call *vertical curve* the set $\Gamma = U \cap F^{-1}(0)$ from Theorem 5.3.7, that is a connected part of $F^{-1}(0)$ localised near 0. We assume, in particular, that

1. Γ enjoys the linear order: $a \leq b \iff z(a^{-1} \cdot b) \geq 0$.
2. There is an increasing function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$, $\alpha(r) \xrightarrow[r\to 0]{} 0$, called modulus of verticality, such that for any $a, b \in \Gamma$

$$\|\pi(a) - \pi(b)\| \leq \alpha(\mathsf{d}(a,b))\mathsf{d}(a,b).$$

   For technical reasons (to start to use the flatness of Γ from its biggest scale) we will require that α is small enough. Certainly, $\alpha(\operatorname{diam}\Gamma) < 1/10$ should be sufficient.

Observe that 1. and 2. imply that (for max-type distance d as in Eq. (2.2))

$$a \leq b \Longrightarrow \mathsf{d}(a,b)^2 = z(a^{-1} \cdot b) \geq 0 \text{ for } a, b \in \Gamma.$$

For $a, b \in \Gamma$ with $a \leq b$, by the interval $[a,b]_\Gamma$ we mean $\{c \in \Gamma \mid a \leq c \leq b\}$.

Remark. In a trivial example, if $\alpha \equiv 0$ then Γ is a part of translated vertical axis.

Remark 5.3.9. By inspecting Eq. (5.4), we note that the modulus of verticality α for $\Gamma \subset F^{-1}(0)$ can be bounded as follows

$$\begin{aligned}\alpha(t) &\lesssim \max_{a\in\Gamma} n_F(\{a\})^{-1}\omega_F(B(a,Ct)) \\ &\lesssim \max_{a\in\Gamma} \frac{\sup_{X\in H\mathbb{H}^1;\,\|X\|=1}\sup_{b\in B(a,Ct)}\|X(a)-X(b)\|}{|\det(\{X_iF(a),Y_iF(a)\mid i=0,\dots,n\})|},\end{aligned}$$

where $C = C(\mathbb{H}^n, \mathsf{d})$ is a geometric constant.

Proposition 5.3.10. *Let Γ be a vertical curve. From Proposition 5.2.12 and Eq. (5.7) we can derive the following relation:*

$$|\mathsf{d}(a,b)^2 + \mathsf{d}(b,c)^2 - \mathsf{d}(a,c)^2| \le 2\alpha^2(2\mathsf{d}(a,c))\mathsf{d}(a,c)^2 \tag{5.12}$$

that holds for $a \le b \le c$ on Γ.

Below we present some simple properties of vertical curve resulting from the flatness.

Proposition 5.3.11. *Let α be a modulus of verticality of vertical curve Γ.*

1. *For every $c \in \mathbb{H}^n$,*
$$\operatorname{diam}(B(c,r)\cap\Gamma) \le \sqrt{2}r(1+\alpha(2r)).$$

2. *For any $a, b \in \Gamma$, the intersection $\bar{B}(a,r)\cap\bar{B}(b,r) = \varnothing$ is empty for*
$$r = \frac{\mathsf{d}(a,b)}{\sqrt{2}}(1-\alpha(\mathsf{d}(a,b))),$$
unless $\alpha \equiv 0$.

3. *For any $a, b \in \Gamma$ with $a \le b$,*
$$0 \le \operatorname{diam}([a,b]_\Gamma) - \mathsf{d}(a,b) \le K\mathsf{d}(a,b)\alpha(K\mathsf{d}(a,b))^2, \quad K = 4.$$

4. *For any $a, b \in \Gamma$, the intersection $\bar{B}(a,r)\cap\bar{B}(b,r)\cap\Gamma = \varnothing$ is empty for*
$$r = \frac{\mathsf{d}(a,b)}{\sqrt{2}}(1-\alpha(2\mathsf{d}(a,b))^2), \tag{5.13}$$
unless $\alpha \equiv 0$.

5. *Any sub-interval $[a,b]_\Gamma \subset \Gamma$ lies insight a ball with centre on $c \in \Gamma$ of radius*
$$r = \frac{\mathsf{d}(a,b)}{\sqrt{2}}(1+K\alpha(K\mathsf{d}(a,b))^2), \quad 0 < K < \infty.$$

Proof. Assume that $a, b \in \Gamma$ with $b \geq a$, then for any $c \in \mathbb{H}^n$ we have

$$\begin{aligned}
\mathsf{d}(a,b)^2 &= z(b) - z(a) - 2\mathcal{B}(\pi(b), \pi(a)) \\
&= \big(z(b) - z(c) - 2\mathcal{B}(\pi(b), \pi(c))\big) - \big(z(a) - z(c) - 2\mathcal{B}(\pi(a), \pi(c))\big) \\
&\quad - 2\big(\mathcal{B}(\pi(b), \pi(a)) - \mathcal{B}(\pi(b), \pi(c)) + \mathcal{B}(\pi(a), \pi(c))\big) \\
&\leq \mathsf{d}(b,c)^2 + \mathsf{d}(a,c)^2 + 2|\mathcal{B}(\pi(b) - \pi(a), \pi(a) - \pi(c))| \\
&= \mathsf{d}(b,c)^2 + \mathsf{d}(a,c)^2 + 2|\mathcal{B}(\pi(b) - \pi(c), \pi(a) - \pi(c))|.
\end{aligned}$$

Note also that $|\mathcal{B}(v, w)| \leq \|v\|\|w\|$. Now we are going to proceed case by case.

1. If $a, b \in B(c, r) \cap \Gamma$ we obtain that $\mathsf{d}(a,b)^2 \leq 2r^2 + 4r^2\alpha(2r)$.

2. If $c \in B(a, r) \cap B(b, r)$, then $\mathsf{d}(a,b)^2 \leq 2r^2 + 2r\mathsf{d}(a,b)\alpha(\mathsf{d}(a,b))$. Solving this quadratic equation one can derive the claimed value of r.

3. It follows from Propositions 5.3.10 and 5.4.2, that $\mathsf{d}(a,b)^2 \geq \operatorname{diam}([a,b])^2\big(1 - 4\alpha(2\operatorname{diam}([a,b]))^2\big)$. Since $\alpha < 1/10$, we have first a rough bound $\operatorname{diam}([a,b]) \leq 2\mathsf{d}(a,b)$, that gives $\operatorname{diam}([a,b])^2 \leq \mathsf{d}(a,b)^2 + 8\mathsf{d}(a,b)^2\alpha(4\mathsf{d}(a,b))^2$. We conclude by taking the square root in this inequality.

4. Assume that $a \leq c \leq b$ on Γ and $\mathsf{d}(a,c) < r$ and $\mathsf{d}(b,c) < r$. Then by Eq. (5.12), $\mathsf{d}(a,b)^2 \leq 2r^2 + 2\mathsf{d}(a,b)^2\alpha(2\mathsf{d}(a,b))^2$, from where we obtain the bound on r.

5. We take $c \in [a,b]_\Gamma$ such that $\mathsf{d}(a,c) = \mathsf{d}(b,c) = \tilde{r}$. Applying Eq. (5.12), we get that $|2\tilde{r}^2 - \mathsf{d}(a,b)^2| \leq 2\mathsf{d}(a,b)^2\alpha(2\mathsf{d}(a,b))^2$. It is clear that $[a,b]_\Gamma \subset B(c,r)$ with $r = \max\{\operatorname{diam}[a,c]_\Gamma, \operatorname{diam}[c,b]_\Gamma\}$. By the previous point, $|r - \tilde{r}| \lesssim \tilde{r}\alpha(\tilde{r})^2$, and the conclusion follows.

□

We give also some elementary consequences of Proposition 5.3.11.

Corollary 5.3.12. *For any interior point $c \in \Gamma$,*

$$\operatorname{diam}(B(c,r) \cap \Gamma) = \sqrt{2}r(1 + o(1)), \quad r \searrow 0, \tag{5.14}$$

where small-o is uniform for $c \in [a', b']_\Gamma$, as soon as $a' > \min\Gamma$ and $b' < \max\Gamma$.

Corollary 5.3.13 (Linear connexity of vertical curves)**.** *Let Γ be a vertical curve with a modulus of verticality α. For $c \in \mathbb{H}^n$ and $r > 0$ such that $\Gamma \cap B(c,r) \neq \varnothing$ we put $a = \min\{\Gamma \cap B(c,r)\}$ and $b = \max\{\Gamma \cap B(c,r)\}$. Then*

$$\max_{c' \in [a,b]_\Gamma} \mathsf{d}(c, c') \lesssim r.$$

Definition. Let (X, d_X) and (Y, d_Y) be two quasi-metric spaces. A homeomorphism $f\colon X \to Y$ is said to be quasisymmetric if there is an increasing function $\eta : [0,\infty) \to [0,\infty)$ such that for any triple a, b, c of distinct points in X, we have

$$\frac{d_Y(f(a), f(b))}{d_Y(f(a), f(c))} \leq \eta\left(\frac{d_X(a,b)}{d_X(a,c)}\right).$$

Observe that the inverse map $f^{-1}\colon Y \to X$ is also quasi-symmetric with $\tilde{\eta} = \eta(t^{-1})^{-1}$.

Proposition 5.3.14. *A vertical curve* (Γ, d) *is quasi-symmetric equivalent to* $([0,1], |\cdot|)$.

This follows from the following metric characterization of quasi-circles.

Theorem ([TV80]). *A metric Jordan curve is quasi-symmetric equivalent to the circle* $\mathcal{S}^1$ *if and only if it is both bounded turning and doubling (that is, of finite Assouad dimension).*

A curve Γ is bounded turning if there is a constant $C > 0$ such that $\operatorname{diam}[a,b]_\Gamma \leq C\mathsf{d}(a,b)$ for any $a, b \in \Gamma$. This property holds for a vertical curve even with $C \to 1$ when $\mathsf{d}(a,b) \to 0$. The Assouad dimension (Definition 3.5.10) of vertical curve is finite merely because it is a subset of $\mathbb{H}^n$ that has finite Assouad dimension. (In fact, the Assouad dimension of (Γ, d) is equal to 2 by Proposition 3.5.12.)

5.4. Flat quasi-metric on interval

Let's understand the properties of Γ that can be derived from Eq. (5.12). We consider here slightly more abstract situation.

Definition 5.4.1. Let κ be a quasi-metric on $[0,1]$ continuous in the standard topology. We assume that κ is *flat* in the following sense:

$$|\kappa(a,b) + \kappa(b,c) - \kappa(a,c)| \leq m(\kappa(a,c))\kappa(a,c), \tag{5.15}$$

holds for $0 \leq a \leq b \leq c \leq 1$ with non-decreasing function $m(t) \searrow 0$ when $t \searrow 0$. We are going to call $\lambda = ([0,1], \kappa)$ *flat curve.*

By Eq. (5.12), the function m for vertical curve Γ endowed with d^2 can be bounded as $m(t) \leq 2\alpha(2\sqrt{t})^2$.

Example. Any Reifenberg flat curve with vanishing constant in $\mathbb{R}^n$ with induced euclidean metric satisfies condition Eq. (5.15), see [DKT01; DT99]. For instance, the flat snowflakes curve (i. e. whose angles decrease with scale) will do. Note that they can have infinite Hausdorff measure $\mathcal{H}^1$.

The following consequence of flatness (that we are going to use frequently) says that at vanishing scales the diameter of an interval on λ is equivalent to the distance between the end-points.

Proposition 5.4.2. *For* $[a,b] \subset \lambda$,

$$\operatorname{diam}([a,b])\big(1 - 2m(\operatorname{diam}([a,b]))\big) \leq \kappa(a,b) \leq \operatorname{diam}([a,b]).$$

Proof. Take $c, d \in [a,b]$, $c \leq d$, such that $\operatorname{diam}([a,b]) = \kappa(c,d)$. Using Eq. (5.15), we conclude by

$$\begin{aligned}
k(a,b) &\geq \kappa(a,b) - \kappa(a,c) - \kappa(d,b) \\
&= (\kappa(a,b) - \kappa(a,d) - \kappa(d,b)) + (\kappa(a,d) - \kappa(a,c) - \kappa(c,d)) + \kappa(c,d) \\
&\geq \kappa(c,d) - m(\kappa(a,b))\kappa(a,b) - m(\kappa(a,d))\kappa(a,d) \\
&\geq \kappa(c,d) - 2m(\kappa(c,d))\kappa(c,d).
\end{aligned}$$

□

Lemma 5.4.3. *There exists a probability measure μ on λ and a constant $0 < \hat{m} < 1$ such that for every $[b, c] \subset [a, b] \subset \lambda$ with $m(\operatorname{diam}([a, b])) \leq \hat{m}$*

$$\left| \ln \left(\frac{\mu([a, b])}{\mu([c, d])} \frac{\kappa(c, d)}{\kappa(a, b)} \right) \right| \leq K \int_{\kappa(c,d)/4}^{4\kappa(a,b)} \frac{m(t)}{t} \, dt, \tag{5.16}$$

where $K = K(\hat{m}) > 0$ is some constant.

Proof. We build (by dyadic iterations) a sequence of families of closed intervals $D^k = \{I_1^k, \ldots, I_{2^k}^k\}$, $k = 0, 1, 2, \ldots$. We put $I_1^0 = [0, 1]$. Assume that we've defined D^k, then each interval $I_i^k, i = 1, \ldots, 2^k$, gives rise to two sub-intervals of D^{k+1} as explained below. If $I_i^k = [a, c]$ then $I_{2i-1}^{k+1} = [a, b]$ and $I_{2i}^{k+1} = [b, c]$ where the point $c \in [a, b]$ is chosen is such a way that $\kappa(a, b) = \kappa(b, c)$ (by the continuity of κ this is always possible). We define probability measure μ on λ by putting $\mu(I) = 2^{-k}$ for any $I \in D^k, k = 0, 1, 2, \ldots$. We put $L(I) := \kappa(a, b)$ for $I = [a, b]$. If $I \in D^n$, we denote by $\bar{I}_k$, $k \leq n$, a unique interval of D^k containing I. If $I \in D_k$ is dyadic, then $d(I) := k$.

Let us take two dyadic intervals $I^s \subset I^b$. Assume that $\operatorname{diam}(I^b) \leq \hat{m} < 1$. According to the flatness condition Eq. (5.15),

$$|L(\bar{I}_{n-1}^s) - 2L(I^s)| \leq m(L(\bar{I}_{n-1}^s))L(\bar{I}_{n-1}^s),$$

so that,

$$1 - m(L(\bar{I}_{d(I^s)-1}^s)) \leq \frac{2L(I^s)}{L(\bar{I}_{d(I^s)-1}^s)} \leq 1 + m(L(\bar{I}_{d(I^s)-1}^s)).$$

Repeating this, we show that

$$\prod_{k=d(I^b)}^{d(I^s)-1} \left(1 - m(L(\bar{I}_k^s))\right) \leq 2^{d(I^b)-d(I^s)} \frac{L(I^s)}{L(I^b)} \leq \prod_{k=d(I^b)}^{d(I^s)-1} \left(1 + m(L(\bar{I}_k^s))\right).$$

By taking the logarithm on this double inequality we obtain

$$\sum_{k=d(I^b)}^{d(I^s)-1} \ln\left(1 - m(L(\bar{I}_k^s))\right) \leq \ln\left(\mu(I)^{-1} \frac{L(I)}{L(\bar{I}_0)}\right) \leq \sum_{k=d(I^b)}^{d(I^s)-1} \ln\left(1 + m(L(\bar{I}_k^s))\right),$$

that gives

$$-K_1(\hat{m}) \sum_{k=d(I^b)}^{d(I^s)-1} m(L(\bar{I}_k^s)) \leq \ln\left(\frac{\mu(I^b)}{\mu(I^s)} \frac{L(I^s)}{L(I^b)}\right) \leq \sum_{k=d(I^b)}^{d(I^s)-1} m(L(\bar{I}_k^s)).$$

Since m is non-decreasing, by classical mean-value theorem

$$\sum_{k=d(I^b)}^{d(I^s)-1} m(L(\bar{I}_k^s)) \leq K_2(\hat{m}) \sum_{k=d(I^b)}^{d(I^s)-1} m(L(\bar{I}_k^s)) \ln\left(\frac{L(\bar{I}_{k-1}^s)}{L(\bar{I}_k^s)}\right) \leq K_2(\hat{m}) \int_{L(I^s)}^{L(I^b)} \frac{m(t)}{t} \, dt,$$

where $K_2(\hat{m}) = \max_{l\leq k<n} \ln(L(\bar{I}^s_{k-1})L(\bar{I}^s_k)^{-1}) \leq -\ln((1+\hat{m})/2)$. Thus, we've proved Eq. (5.16) for dyadic intervals.

Let us next prove Eq. (5.16) for a dyadic interval $I^b \in D_l$ and an arbitrary interval $I^s \subset I^b$ (compare the arguments with [Kor98, Th. 2]). General case can be reduced to this one. Indeed, if I^b is not dyadic, we can find a dyadic interval $\tilde{I}^b$ such that $I^b \subset \tilde{I}^b$ and $4\mu(I^b) > \mu(\tilde{I}^b)$. Then we should apply our estimates twice: once for the distortion between $\tilde{I}^b$ and I^b, and a second time, for $\tilde{I}^b$ and I^s.

Let $I^s = \bigcup_{i\geq 1} I^k$ be the unique decomposition of I^s into maximal dyadic intervals $\{I^k\}$ ordered according to their size. Note that the largest interval $I^1 \in D_n$ satisfies $\mu(I^1) \geq \mu(I^s)/4$. Again recursively, using the flatness of λ, we can obtain the estimate (with convention $I_0 = I^s$)

$$\sum_{k\geq 1} L(I^k) \prod_{i=0}^{k-1} \big(1+m(L(I_i))\big)^{-1} \leq L(I^s) \leq \sum_{k\geq 1} L(I^k) \prod_{i=0}^{k-1} \big(1-m(L(I_i))\big)^{-1}.$$

By the estimate for dyadic intervals, we get an upper bound

$$\begin{aligned}
\frac{\mu(I^b)}{\mu(I^s)}\frac{L(I^s)}{L(I^b)} &\leq \sum_{k\geq 1} \frac{\mu(I^k)}{\mu(I^s)}\frac{\mu(I^b)}{\mu(I^k)}\frac{L(I^k)}{L(I^b)} \prod_{i=0}^{k-1} \big(1-m(L(I_i))\big)^{-1} \\
&\leq \sum_{k\geq 1} \frac{\mu(I^k)}{\mu(I^s)} \prod_{l=d(I^b)}^{d(I^k)-1} \big(1+m(L(\bar{I}^k_l))\big) \prod_{i=0}^{k-1} \big(1-m(L(I_i))\big)^{-1} =: P(1+R),
\end{aligned}$$

where

$$P = \prod_{l=d(I^b)}^{d(I^1)-1} \big(1+m(L(\bar{I}^1_l))\big),$$

and

$$R = \sum_{k\geq 2} \frac{\mu(I^k)}{\mu(I^s)} \left(\prod_{l=d(I^1)}^{d(I^k)-1} \big(1+m(L(\bar{I}^k_l))\big) \prod_{i=0}^{k-1} \big(1-m(L(I_i))\big)^{-1} - 1 \right).$$

The factor P is good and can be bounded as before, so, let us deal with R.

Observe that by the maximality of the dyadic family $\{I_k\}$, $d(I^k) - d(I^1) \geq k/2$. Obviously, $\mu(I^k)/\mu(I^s) \leq 2^{d(I^1)-d(I^k)}$. Since m is non-decreasing,

$$R \leq \sum_{k\geq 2} \frac{\mu(I^k)}{\mu(I^s)} \left(\frac{(1+m(L(I^s)))^{d(I^k)-d(I^1)}}{(1-m(L(I^s)))^k} - 1 \right) \leq \sum_{k\geq 2} 2^{d(I^1)-d(I^k)} \left(q^{d(I^k)-d(I^1)} - 1 \right),$$

where

$$q := \frac{1+m(L(I^s))}{(1-m(L(I^s)))^2}.$$

Assume that I^s is small enough in such a way that $q \leq q_0 < 2$. Then, using the fact that $(1+t)^l - 1 \leq lt(1+t_0)^{l-1}$ if $t \in [0, t_0]$ and the convergence of geometric series, we get that

$$R \leq (q-1)\sum_{k\geq 2}(d(I^k)-d(I^1))\left(\frac{q_0}{2}\right)^{d(I^k)-d(I^1)-1} \leq K_3(\hat{m})m(L(I^s)).$$

Thus, we obtain an appropriate upper bound. The lower bound can be obtained in exactly the same way. □

Remark 5.4.4. Observe that a measure μ in Lemma 5.4.3 is not canonical and there is some flexibility in its definition (unless $m \equiv 0$). Looking at the proof, we observe that two new subintervals I_1 and I_2 inside $I = I_1 \cup I_2$ must satisfy

$$1 - m(L(I)) \leq 2L(I_k)L(I)^{-1} \leq 1 + m(L(I)), \quad k = 1, 2,$$

in order to obtain the measure μ with required properties. This means that each time we can move a middle point $c = I_1 \cap I_2$ by a distance $\lesssim m(L(I))L(I)$.

Thus, returning to the vertical curves (Γ, d^2), for each interval $I = [a,b]_\Gamma$ we can chose as a middle point any point $c \in I \setminus (B(a,r) \cup B(b,r))$, where r as defined in Eq. (5.13) (note that $\alpha(2\mathsf{d}(a,b))^2 \approx m(L(I))$ by Eq. (5.12)).

Corollary 5.4.5. *Measure μ is asymptotically optimally doubling on λ (see [DKT01]) in the following sense: for any fixed $C \geq 2$*

$$\lim_{\delta\to 0} \sup_{\kappa(a,b)<\delta} \left\{\left|\frac{\mu([a,b])}{\mu([c,d])} - \frac{\kappa(a,b)}{\kappa(c,d)}\right|; \quad [c,d] \subset [a,b] \subset \lambda, \quad \kappa(a,b) \leq C\kappa(c,d)\right\} = 0.$$

Proof. Indeed, for $\delta \to 0$ we have

$$\int_{\kappa(c,d)/4}^{4\kappa(a,b)} \frac{m(t)}{t}\,dt \leq m(4\kappa(a,b)) \ln\left(16\frac{\kappa(a,b)}{\kappa(c,d)}\right) \leq m(4\delta)\ln(16C) \to 0,$$

so that,

$$\left|\frac{\mu([a,b])}{\mu([c,d])} - \frac{\kappa(a,b)}{\kappa(c,d)}\right| \leq C\left|\frac{\mu([a,b])}{\mu([c,d])}\frac{\kappa(c,d)}{\kappa(a,b)} - 1\right| \to 0.$$

□

Below we give some useful consequences of the existence of measure μ on λ.

Corollary 5.4.6 (bi-Hölder parametrization)**.** *For any $\epsilon > 0$ there is a constant $K = K(m, \epsilon, \mathrm{diam}(\lambda)) \geq 1$ such that for every $[a,b] \subset \lambda$*

$$K^{-1}\mu([a,b])^{1+\epsilon} \leq \kappa(a,b) \leq K\mu([a,b])^{1-\epsilon}. \tag{5.17}$$

Proof. For instance, for an upper bound we have

$$\frac{\kappa(a,b)}{\mu([a,b])^{1-\epsilon}} \lesssim \kappa([a,b])^{\epsilon} \exp\left(\int_{\kappa(a,b)/4}^{\mathrm{diam}\,\lambda} \frac{m(t)}{t}\,dt\right)^{1-\epsilon} = \exp\left(\int_{\kappa(a,b)/4}^{\mathrm{diam}\,\lambda} \frac{m(t)-l}{t}\,dt\right)^{1-\epsilon},$$

where

$$l = \frac{\epsilon \ln(\kappa(a,b))}{(1-\epsilon)(\ln(\kappa(a,b)) - \ln(4\operatorname{diam}\lambda))} \geq \frac{\epsilon}{2(1-\epsilon)},$$

provided that $\kappa(a,b)^2 \leq 4\operatorname{diam}\lambda$. Since $m \searrow 0$, there is some moment $t^\star > 0$ such that $m(t) - \dfrac{\epsilon}{2(1-\epsilon)} \leq 0$ for all $t \leq t^\star$, and, hence,

$$K \lesssim \exp\left(\int_{t^\star}^{\operatorname{diam}\lambda} \frac{m(t)-l}{t}\,dt\right)^{1-\epsilon}.$$

□

Corollary 5.4.7. *Every flat curve λ has finite p-variation, $\operatorname{Var}^p \lambda < \infty$, for any $p > 1$ and infinite p-variation $\operatorname{Var}^p \lambda = \infty$ if $p < 1$.*

Corollary 5.4.8. *The Hausdorff dimension of the flat curve λ is equal to 1.*

Proof. It follows from the mass distribution principle, see [Fal03; Fed69], because by Proposition 5.4.2 and Eq. (5.17)

$$\lim_{\kappa(a,b)\to 0} \mu([a,b])\operatorname{diam}([a,b])^{-\alpha} = \lim_{\kappa(a,b)\to 0} \mu([a,b])\kappa(a,b)^{-\alpha} = \begin{cases} 0, & \text{if } \alpha > 1; \\ \infty, & \text{if } \alpha < 1. \end{cases}$$ □

Lemma 5.4.9 (Area formula)**.** *For the flat curve λ, the 1-dimensional Hausdorff measure can be computed by the formula*

$$\mathcal{H}^1(\lambda) = \liminf_{\delta\to 0} \sum_{i=0}^{l} \kappa(a_i, a_{i+1}), \tag{5.18}$$

where the infimum is taken over all subdivisions $\sigma = \{0 = a_0 < a_1 < \ldots < a_l < a_{l+1} = 1\}$ such that $\|\sigma\| := \max_{i=0,\ldots,l} |a_i - a_{i+1}| < \delta$.

Proof. Let's denote the right hand side of Eq. (5.18) by $T(\lambda)$. By Proposition 5.4.2, $T(\lambda) = \liminf_{\delta\to 0} \sum_{i=0}^{l} \operatorname{diam}([a_i, a_{i+1}])$. So, it is clear that $\mathcal{H}^1(\lambda) \leq T(\lambda)$ because coverings by intervals are more restricted than those in the definition of Hausdorff measure. Let us assume that $\mathcal{H}^1(\lambda) < \infty$ and show the inverse inequality.

Let $\lambda \subset \bigcup_{i=0}^{N} E_i$, $0 < \operatorname{diam}(E_i) < \epsilon$, E_i is open, finite (due to the compactness of λ) covering of λ. Thanks to the continuity of κ, we can consider only the coverings by open sets. For $E_i \neq \varnothing$ we define $a_i = \inf\{E_i\}$ and $b_i = \sup\{E_i\}$. Obviously, $\lambda \subset \bigcup_i [a_i, b_i]$. Then, we can find a sequence of points $0 = c_0 < c_1 < \ldots < c_l < c_{l+1} = 1$ such that any interval $[c_i, c_{i+1}]$ lies in some $[a_k, b_k]$ and two successive intervals $[c_i, c_{i+1}]$ and $[c_{i+1}, c_{i+2}]$ don't belong to the same $[a_k, b_k]$. By Proposition 5.4.2,

$$\operatorname{diam}([a_i, b_i]) \leq \kappa(a_i, b_i)(1 + 2m(\epsilon)) \leq \operatorname{diam}(E_i)(1 + 2m(\epsilon)).$$

Since $m(\epsilon) \to 0$ when $\epsilon \to 0$, we conclude by

$$\sum_{i=0}^{l} \kappa(c_i, c_{i+1}) \leq \sum_k \operatorname{diam}([a_k, b_k]) \leq (1 + 2m(\epsilon)) \sum_k \operatorname{diam}(E_k).$$ □

Example. Consider quasi-metrics of the form $\kappa(a,b) = |a-b|\phi(|a-b|)$ with some positive smooth function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$. We want to see when κ is flat in the sense of Eq. (5.15). So, $0 \le s \le t$, we need to have that

$$\frac{|t\phi(t) + s\phi(s) - (t+s)\phi(t+s)|}{(t+s)\phi(t+s)} \to 0, \quad t \to 0.$$

By mean value theorem, $|t\phi(t) + s\phi(s) - (t+s)\phi(t+s)| \le 2ts\|\dot{\phi}\|_{[s,t+s]}$. Hence, it suffices that $s\|\dot{\phi}\|_{[s,t+s]}/\phi(t+s) \to 0$ when $t \to 0$. This is the case, for instance, for $\phi_1(t) = 1 + |\ln(t)|$, and $\phi_2(t) = \phi_1(t)^{-1}$ for which we have

$$\begin{aligned} s\|\dot{\phi}_1\|_{[s,t+s]}\, \phi_1(t+s)^{-1} &\lesssim |\ln(t+s)|^{-1} \to 0, \\ s\|\dot{\phi}_2\|_{[s,t+s]}\, \phi_2(t+s)^{-1} &\lesssim |\ln(s)|^{-1} \to 0. \end{aligned}$$

Note that by Eq. (5.18) the corresponding quasi-metrics satisfy

$$\mathcal{H}^1_{\kappa_1}([0,1]) = \infty; \quad \mathcal{H}^1_{\kappa_2}([0,1]) = 0.$$

Question 5.4.10. Can $([0,1],\kappa_1)$ or $([0,1],\kappa_2)$ be bi-Lipschitz equivalent to squared metric (Γ, d^2) induced on some vertical curve $\Gamma \Subset \mathbb{H}^n$? In general, one could think about some intrinsic characterization (up to bi-Lipshitz equivalence) of metrics coming as an induced metric on vertical curves (as, for instance, in [Roh01; HM12] for quasi-circles).

Definition 5.4.11. The flat curve λ is said to be p-*Ahlfors* regular if there is $1 \le K < \infty$ such that

$$K^{-1}\kappa(a,b)^p \le \mathcal{H}^p([a,b]) \le K\kappa(a,b)^p$$

holds for any $[a,b] \subset \lambda$.

Remark. If λ is p-Ahlfors regular, then

$$0 < K^{-1}\mathcal{H}^p(\lambda) \le \mathrm{Var}^p(\lambda) \le K\mathcal{H}^p(\lambda) < \infty.$$

Lemma 5.4.12 (Area formula in regular case)**.** *Assume that m satisfies Dini's condition*

$$D(\lambda) := \int_0^{\mathrm{diam}(\lambda)} \frac{m(t)}{t}\,dt < \infty. \tag{5.19}$$

The flat curve λ is 1-Ahlfors regular in a strong sense: for any $[s,t] \subset \lambda$

$$\begin{aligned} \mathcal{H}^1([s,t]) &= \lim_{\|\sigma\|\to 0} \sum_{k=0}^{l} \kappa(t_k, t_{k+1}), \quad \sigma = \{s = t_0 < t_1 < \cdots < t_{l+1} = t\}, \\ |\mathcal{H}^1([s,t]) - \kappa(s,t)| &\le C\kappa(s,t) \int_0^{C\kappa(s,t)} \frac{m(t)}{t}\,dt, \end{aligned} \tag{5.20}$$

where $C = C(D(\lambda)) < \infty$.

Remark. Dini condition $\int_0^* \frac{m(t)}{t}\,dt < \infty$ is equivalent to the summability of $\sum_k m(r^{-k})$ for any $1 < r < \infty$.

Proof. Assume first that $m(\operatorname{diam}(\lambda)) < \hat{m}$ as in Lemma 5.4.3. We can always achieve this condition by taking small pieces of λ and loosing some constants in our estimates.

Since $D(\lambda) < \infty$, from Eq. (5.16) we see that the ratio $R_0/C < \kappa(c,d)/\mu([c,d]) < CR_0$ is bounded from above and from below when $\kappa(c,d) \to 0$, where $C = \exp(D(\lambda) + m(\operatorname{diam}(\lambda))) + K$ and $R_0 = \operatorname{diam}(\lambda)/\mu(\lambda)$. In particular, λ is 1-Ahlfors regular.

We can chose a bi-Lipschitz parametrization $([0,T], |\cdot|) \longrightarrow ([0,1], \kappa)$ in such a way that $|t-s|/C^2 \le \kappa(s,t) \le |t-s|$ for $[s,t] \subset [0,T]$. So, the flatness condition Eq. (5.15) will read

$$|\kappa(s,t) + \kappa(h,t) - \kappa(s,h)| \le m(|s-t|)|s-t|, \quad h \in [s,t] \subset [0,T].$$

To obtain the "regular" version of the area formula we should apply the following generalization of a classical result on Stieltjes integration (see, also [You38; Bur48]).

Lemma 5.4.13 (about additive sewing, [FL06])**.** *Let $\kappa : [0,1]^2 \to \mathbb{R}$ be a continuous function such that*

$$|\kappa(a,b) + \kappa(b,c) - \kappa(a,c)| \le \omega(|a-c|), \quad b \in [a,c],$$

with a non-decreasing function $\omega(t)$. Suppose that $\sum_{i=0}^{\infty} 2^i \omega(2^{-i}) < \infty$. Then there exists a unique (up to additive constant) function $\nu : [0,1] \to \mathbb{R}$ such that

$$|\nu(b) - \nu(a) - \kappa(a,b)| \le \sum_{i=0}^{\infty} 2^i \omega(|b-a|2^{-i}).$$

Furthermore, the Stieltjes sums $\sum_{i=0}^{l} \kappa(t_i, t_{i+1})$, where $\sigma = \{a = t_0 < t_1 < \cdots < t_{l+1} = b\}$ is a subdivision of $[a,b]$, converges to $\nu(b) - \nu(a)$ when $\|\sigma\| \to 0$.

In our case, $\omega(t) = tm(t)$, so that

$$\sum_{i=0}^{\infty} 2^i \omega(|b-a|2^{-i}) = |b-a| \sum_{i=0}^{\infty} m(|b-a|2^{-i}) \le C^2 \kappa(a,b) \sum_{i=0}^{\infty} m(C^2 \kappa(a,b) 2^{-i}),$$

and Eq. (5.20) follows. □

Remark. Observe that the doubling measure μ on λ is obtained by "top-bottom" scale procedure whereas the Hausdorff measure $\mathcal{H}^1$ by "bottom-top" one. We see that under Dini's condition Eq. (5.19), these two measures of different nature are comparable.

Elements of quantitative analysis. Assume that the flat curve λ lives at scale $N \in \mathbb{Z}$, i. e. $\operatorname{diam}(\lambda) \approx 2^{-N}$. Let

$$\mathcal{A}_N \subset \mathcal{A}_{N+1} \subset \mathcal{A}_{N+2} \subset \dots$$

be a filtration of nets on λ such that $\mathcal{A}_N = \{0,1\}$ and $\mathcal{A}_n$, $n > N$, is 2^{-n}-scale skeleton on λ:

- for each $a \in \lambda$ there is $b \in \mathcal{A}_n$ such that $\kappa(a,b) \le 2^{-n}$;

- $\kappa(b, b') \geq 2^{-n+1}$ for any couple of different points $b, b' \in \mathcal{A}_n$.

There is no issue for the construction of a such filtration.

Each skeleton inherits the order structure from λ, so it can be seen as a subdivision of λ. To a subdivision $\{0 = a_1 < \ldots < a_{k+1} = 1\}$ we naturally associate an ordered set of covering intervals $\{[a_i, a_{i+1}] \mid i = 1, \ldots, k\}$. So, let $\mathcal{I}_n$ be the set of covering intervals associated to $\mathcal{A}_n$. We define a tree $\mathbb{T}$ encoding the hierarchy of the intervals in this filtration:

- the set of nodes of $\mathbb{T}$ is $\bigsqcup_{n\geq 0} \mathcal{I}_n$;
- an interval $I \in \mathcal{I}_n$ is the parent of I' if and only if $I' \in \mathcal{I}_{n+1}$ and $I' \subset I$.

The symbol $I^{\pitchfork}$ stays for the set of children of the node $I \in \mathbb{T}$. Note that for all $I \in \mathbb{T}$ the number of children $\#(I^{\pitchfork}) < C$ is uniformly bounded. For $I \in \mathbb{T}$ we introduce

$$\partial(I) := \sum_{I' \in I^{\pitchfork}} \operatorname{diam}(I') - \operatorname{diam}(I).$$

For any interval $I \subset \lambda$, we denote by $\mathbb{T}_I$ a sub-tree of $\mathbb{T}$ whose root is the smallest interval $I' \in \mathbb{T}$ containing I. For a tree, we call the *cut* (of variable depth) a set of leaves of some finite sub-tree growing from the same root.

Lemma 5.4.14. *Assume that $\sum_{I\in\mathbb{T}} |\partial(I)| < \infty$. Then the flat curve λ satisfies*

$$|\mathcal{H}^1(I) - \operatorname{diam}(I)| \lesssim m(\operatorname{diam}(I)) \operatorname{diam}(I) + \sum_{I' \in \mathbb{T}_I} |\partial(I')| \tag{5.21}$$

for any interval $I \subset \lambda$.

Proof. Let $I \in \mathbb{T}$, then for any cut $cut(\mathbb{T}_I)$ of $\mathbb{T}_I$ we have that

$$|\operatorname{diam}(I) - \sum_{I' \in cut(\mathbb{T}_I)} \operatorname{diam}(I')| \leq \sum_{I \in \mathbb{T}_I} |\partial(I)|.$$

By density of "dyadic" points and the area formula, there is a sequence σ_n of cuts of $\mathbb{T}_I$ with $\max_{I' \in \sigma_n} \operatorname{diam}(I') \xrightarrow[n \to 0]{} 0$ such that

$$\mathcal{H}^1(I) = \lim_{n\to\infty} \sum_{I' \in \sigma_n} \operatorname{diam}(I').$$

Thus, Eq. (5.21) holds for any $I \in \mathbb{T}$ (even without first term).

Take now arbitrary interval $I \subset \lambda$. This interval can be represented as $I = \bigcup_{I' \in \sigma} I'$ a (disjoint) union of dyadic intervals from $\mathbb{T}$. Therefore,

$$|\mathcal{H}^1(I) - \operatorname{diam}(I)| \leq |\operatorname{diam}(I) - \sum_{I' \in \sigma} \operatorname{diam}(I')| + \sum_{I' \in \sigma} |\mathcal{H}^1(I') - \operatorname{diam}(I')|.$$

For the second term we already have a good estimate:

$$\sum_{I'\in\sigma} |\mathcal{H}^1(I') - \operatorname{diam}(I')| \le \sum_{I'\in\sigma} \sum_{I''\in\mathbb{T}_{I'}} |\partial(I'')| \le \sum_{I''\in\mathbb{T}_I} |\partial(I'')|.$$

For the first term we should use the flatness of λ. We apply recursively Eq. (5.15) to the intervals $I' \in \sigma$ ordered according to their sizes and due to the fact that those sizes are exponentially decreasing we can get that

$$|\operatorname{diam}(I) - \sum_{I'\in\sigma} \operatorname{diam}(I')| \lesssim m(\operatorname{diam}(I)) \operatorname{diam}(I).$$

This argument is very similar to the one at end of proof of Lemma 5.4.3, so the reader should have no difficulties reconstruct it. □

Remark. By Proposition 5.4.2, Eq. (5.21) is equivalent to

$$|\mathcal{H}^1((s,t)) - \kappa(s,t)| \lesssim m(\operatorname{diam}((s,t))) \operatorname{diam}((s,t)) + \sum_{I'\in\mathbb{T}_{(s,t)}} |\partial(I')|.$$

5.4.1. Application to vertical curves. Area formula

We return to the properties of vertical curves. Let Γ be a vertical curve as in Notation 5.3.8. Let $\sigma = \{a = a_0 < a_1 < \cdots < a_l < a_{l+1} = b\}$ be a subdivision of $\Gamma = [a,b]_\Gamma$. We see that

$$\begin{aligned}\sum_{i=0}^{l} \mathsf{d}(a_i, a_{i+1})^2 &= \sum_{i=0}^{l} z_{i+1} - z_i + 2\mathcal{B}(\pi(a_i), \pi(a_{i+1})) \\ &= z(b) - z(a) + 2\sum_{j=1}^{n}\sum_{i=0}^{l} x_i^j(y_{i+1}^j - y_i^j) - y_i^j(x_{i+1}^j - x_i^j),\end{aligned}$$

where x_i^j (y_i^j or z_i) is x^j (y^j or z) coordinate of a_i. Observe that if the image of $\gamma \in C^0([t_i, t_{i+1}], \mathbb{R}^2)$ is a linear segment then

$$\int_\gamma x\, dy = \gamma_x(t_i)(\gamma_y(t_{i+1}) - \gamma_y(t_i)),$$

so, it makes sense to use a piece-wise linear approximation of Γ.

Definition 5.4.15. Let $\sigma = \{a = a_0 < \cdots < a_n = b\}$ be a subdivision of $\Gamma = [a,b]_\Gamma$. To σ we associate an approximating curve Γ_σ such that Γ_σ coincides with Γ on σ and Γ_σ is a linear segment (in $\mathbb{R}^{2n+1}$) between two consecutive points of σ.

According to Eq. (5.12), (Γ, d^2) is a flat curve. Since Hausdorff measure is invariant under isometric embeddings, we derive from Corollary 5.4.8 and Lemma 5.4.9 the following

Corollary 5.4.16. *Vertical curve* $\Gamma \subset \mathbb{H}^n$ *has Hausdorff dimension* 2, $\dim \Gamma = 2$*, and two-dimensional Hausdorff measure of* Γ *can be computed by the formula*

$$\begin{aligned}\mathcal{H}^2(\Gamma) &= z(b) - z(a) + 2 \liminf_{\|\sigma\| \to 0} \sum_{j=1}^{n} \sum_{i=0}^{l} (x_i^j y_{i+1}^j - y_i^j x_{i+1}^j) \\ &= \int_\Gamma dz + 2 \liminf_{\|\sigma\| \to 0} \sum_{j=1}^{n} \int_{\Gamma_\sigma} (x^j\, dy^j - y^j\, dx^j).\end{aligned} \tag{5.22}$$

We can also formulate the area formula in term of the approximating curves.

Proposition 5.4.17. Γ_σ *is also vertical curve with* $\mathcal{H}^2(\Gamma_\sigma) = \sum_\sigma \mathsf{d}(a_{i+1}, a_i)^2$*. Furthermore,* Γ_σ *converges to* Γ *in the sense of Hausdorff distance when* $\|\sigma\| \to 0$ *and, by the area formula,*

$$\liminf_{\|\sigma\| \to 0} \mathcal{H}^2(\Gamma_\sigma) = \mathcal{H}^2(\Gamma).$$

On the contrary, the spherical Hausdorff measure of subsets depends in general on the ambient space.

Proposition 5.4.18. *For our choice of the metric* d *on* $\mathbb{H}^n$ *(see Eq. (2.2)) the spherical Hausdorff measure satisfies*

$$\mathcal{S}^2 \llcorner \Gamma = 2\, \mathcal{H}^2 \llcorner \Gamma.$$

Proof. We need to show two inequalities. We shall use the basic properties of vertical curves from Proposition 5.3.11.

- On one hand, $\operatorname{diam}(B(c,r) \cap \Gamma) \leq \sqrt{2} r(1 + o(1))$, where small-$o$ goes to 0 with $r \to 0$ uniformly w.r.t. $c \in \mathbb{H}^n$.
- On the other hand, any interval $[a, b] \subset \Gamma$ lies inside a ball of radius $r = \mathsf{d}(a,b)/\sqrt{2}(1+o(1))$, with small-$o$ going to 0 when $a \to b$ uniformly w.r.t. $a, b \in \Gamma$.

□

Remark 5.4.19. Imagine that instead of d, we consider another left-invariant homogeneous quasi-metric $\hat{\mathsf{d}}$ on $\mathbb{H}^n$. We will have

$$\mathcal{H}^2_{\hat{\mathsf{d}}}(\Gamma) = c(\hat{\mathsf{d}}) \mathcal{H}^2(\Gamma),$$

where $c(\hat{\mathsf{d}}) = \hat{\mathsf{d}}(0, \exp(Z)(0))/\mathsf{d}(0, \exp(Z)(0))$ is the coefficient of dilatation along the vertical axis of $\hat{\mathsf{d}}$ w.r.t. our reference metric d. Indeed, because of the Whitney's condition, $\operatorname{diam}_{\hat{\mathsf{d}}}(E) = c(\hat{\mathsf{d}}) \operatorname{diam}(E)(1 + o(1))$ when $\operatorname{diam}(E) \to 0$ for $E \subset \Gamma$ and the conclusion follows easily.

5.5. Regular case

Definition 5.5.1. We say that the vertical curve Γ is *strongly Ahlfors regular* (or ω-regular), if there is a modulus ω, such that $\omega(t) \searrow 0$ when $t \to 0$ and

$$|\mathcal{H}^2([c,d]_\Gamma) - \mathsf{d}(c,d)^2| \leq \omega(\mathsf{d}(c,d)^2)\, \mathsf{d}(c,d)^2, \quad [c,d]_\Gamma \subset \Gamma.$$

Remark. This definition is a counterpart of the notion of "vanishing chord-arc" (see, for instance, [CKL05, p. 10]).

Notation 5.5.2. For a curve $\gamma \in C^0([0,T], \pi(\mathbb{H}^n) \equiv \mathbb{R}^{2n})$ we introduce "symplectic" Stieltjes integral

$$\mathrm{Z}(\gamma) := 2 \int_\gamma \sum_{j=1}^{n} (x^j\, dy^j - x^j\, dx^j) = 2 \lim_{\|\sigma\| \to 0} \sum_{i=0}^{l} \mathcal{B}(\gamma(t_i), \gamma(t_{i+1})),$$

where $\sigma = \{0 = t_0 < \cdots < t_{l+1} = T\}$ is a subdivision of $[0,T]$. Remark that the existence of $\mathrm{Z}(\gamma)$ does not imply in general the existence of any of single integrals $\int_\gamma x^j\, dy^j$, so we cannot bring the sum over $j = 1, \ldots, n$ out of the limit over σ.

We give here an immediate consequence of the definitions.

Corollary 5.5.3. *Assume that the vertical curve $\Gamma \Subset \mathbb{H}^n$ is ω-regular. Then the area formula reads*

$$\mathcal{H}^2(\Gamma) = \int\limits_\Gamma dz + \mathrm{Z}(\pi(\Gamma)).$$

The curve Γ admits a natural parametrization $t \to \Gamma(t)$ such that

$$\mathcal{H}^2\big(\Gamma([0,t])\big) = t, \quad t \in [0, \mathcal{H}^2(\Gamma)].$$

This map, $t \to \Gamma(t)$, is bi-Hölder of exponent $1/2$:

$$K(|t-s|)|t-s|^{\frac{1}{2}} \geq \mathsf{d}(\Gamma(t), \Gamma(s)) \geq K(|t-s|)^{-1}|t-s|^{\frac{1}{2}},$$

with $K(\delta) \to 1$ when $\delta \to 0$, and it has the following form

$$t \longrightarrow \big(\pi(\Gamma \llcorner [0,t]), z(0) + t - \mathrm{Z}(\pi(\Gamma \llcorner [0,t]))\big). \tag{5.23}$$

Remark 5.5.4. If Γ is ω-regular vertical curve in $\mathbb{H}^1$, then two integrals $\int_\Gamma x^1\, dy^1$ and $\int_\Gamma y^1\, dx^1$ exist individually. Indeed, w. r. t. natural parametrization the coordinates x^1 and y^1 along Γ belong to $\mathrm{hol}^{1/2}$ (by Whitney's condition). By Remark A.1.4 integration by parts is possible, that implies the existence of

$$2 \int_\Gamma x^1\, dy^1 = x^1 y^1\big|_\Gamma + \int\limits_\Gamma (x^1\, dy^1 - y^1\, dx^1).$$

For higher Heisenberg groups this property is not necessary valid.

In general, vertical curve Γ is not ω-regular (see Section 5.6). However, an additional regularity of the map F assures its level set be ω-regular.

Lemma 5.5.5. *Suppose that the horizontal differential of F has a modulus of continuity m,*

$$\|D_h F(a)^{-1} \cdot D_h F(b)\| \leq \alpha_F(\mathsf{d}(a,b)),$$

with non-decreasing $\alpha_F \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that α_F^2 satisfies Dini's condition Eq. (5.19). Then any vertical curve $\Gamma \subset F^{-1}(0)$ is ω-regular with

$$w(t) \leq K \int_0^{K\sqrt{t}} \frac{\alpha_F(s)^2}{s}\, ds, \quad K < \infty.$$

(Of course, we suppose that $D_h F$ is surjective on Γ.)

Proof. By Remark 5.3.9, the modulus of verticality α of Γ can be controlled by α_F. We can apply Eq. (5.20) with a function m such that $m(s) \lesssim \alpha_F(\sqrt{s})^2$ by Proposition 5.3.10. Thus, for any $a, b \in \Gamma$ we have that

$$\frac{|\mathcal{H}^2([a,b]_\Gamma) - \mathsf{d}(a,b)^2|}{\mathsf{d}(a,b)^2} \leq K \int_0^{K\mathsf{d}(a,b)^2} \frac{\alpha_F(\sqrt{s})^2}{s}\, ds = 2K \int_0^{\sqrt{K}\mathsf{d}(a,b)} \frac{\alpha_F(s)^2}{s}\, ds.$$

□

Corollary 5.5.6. *If $F \in C_h^{1,\alpha}(\mathbb{H}^n, \mathbb{R}^{2n})$, $\alpha > 0$, then its level set $\Gamma \subset F^{-1}(0)$ is ω-regular with $\omega(t) \lesssim t^\alpha$.*

Remark. In the first Heisenberg group $\mathbb{H}^1$, one can show that if α_{F_1} satisfies Dini's condition for only one of the coordinate functions of $F = (F_1, F_2)$ then Γ is ω-regular.

Question 5.5.7. Assume that Γ is ω-regular. Does this imply that a doubling measure μ (constructed in Lemma 5.4.3) is comparable to $\mathcal{H}^2$ on Γ?

5.5.1. Regular vertical lift

A straightforward consequence of Theorem A.1.3 is

Proposition. *Let $\gamma \in \mathrm{Hol}^\alpha([0,T], \mathbb{R}^{2n})$, $\alpha > 1/2$. Then the "horizontally" lifted curve*

$$t \to (\gamma(t), -\mathrm{Z}(\gamma \llcorner [0,t])) \in \mathrm{Hol}^\alpha([0,T], \mathbb{H}^n) \cap \mathrm{Hol}^\alpha([0,T], \mathbb{R}^{2n+1}) \tag{5.24}$$

is Hölder continuous in both Carnot and Euclidean metrics. Conversely, if a curve

$$\{t \to (\gamma(t), z(t))\} \in \mathrm{Hol}^\alpha([0,T], \mathbb{H}^n), \quad \alpha > \frac{1}{2},$$

then for all $t \in [0,T]$

$$z(t) = z(0) - \mathrm{Z}(\gamma \llcorner [0,t]).$$

Remark. There exists also a (non-unique and non-canonical) lift, see [LV07, Pr. 3], of any Hölder curve $\gamma \in \mathrm{Hol}^\alpha([0,T],\mathbb{R}^{2n})$, $\alpha < \frac{1}{2}$, to a Hölder map into $\mathbb{H}^n$, that is a curve $(\gamma, z) \in \mathrm{Hol}^\alpha([0,T],\mathbb{H}^n)$.

Proposition 5.5.8. *Let $\{t \to (\gamma(t), z(t))\} \in \mathrm{Hol}^{1/2}([0,1],\mathbb{H}^n)$ be a Hölder curve such that $\gamma \in \mathrm{hol}^{1/2}([0,1],\mathbb{R}^{2n})$. Then there is a constant $K > 0$ such that*

$$\Gamma = \{t \to (\gamma(t), z(t) + Kt)\}$$

is a vertical 2-Ahlfors regular curve.

Proof. For all $t, s \in [0,1]$

$$|z(\Gamma^{-1}(s) \cdot \Gamma(t))| = |z(t) - z(s) + 2\mathcal{B}(\gamma(s), \gamma(t)) + K(t-s)|,$$

so, by triangle inequality,

$$(K - C)|t-s| \le |z(\Gamma^{-1}(s) \cdot \Gamma(t))| \le (K + C)|t - s|,$$

where $C = \|(\gamma, z)\|^2_{\mathrm{Hol}^{1/2}}$. Put $K = 2C$. Because $\gamma \in \mathrm{hol}^{1/2}$, the following estimate holds uniformly

$$\|\pi(\Gamma(s)^{-1} \cdot \Gamma(s))\|^2 = \|\gamma(t) - \gamma(s)\|^2 = o(|t-s|) = o(|z\big(\Gamma(s)^{-1} \cdot \Gamma(t)\big)|).$$

We see then that curve Γ satisfies Whitney's condition (see Proposition 5.2.5), and, so, it is vertical. Note also that (Γ, d) is bi-Lipschitz equivalent to $([0,1], |\cdot|^{\frac{1}{2}})$, this is where 2-Ahlfors regularity comes from. $\square$

Definition 5.5.9. Let $\gamma \in C^0([0,T],\mathbb{R}^{2n})$ be a curve such that $\mathrm{Z}(\gamma)$ exists. We call *regular vertical lift* of γ a curve of the following form:

$$[0,T] \ni t \to \big(\gamma(t), C - \mathrm{Z}(\gamma \llcorner [0,t]) + t\big).$$

Remark 5.5.10. If $\gamma \in \mathrm{Hol}^\alpha([0,T],\mathbb{R}^{2n})$, $1 \ge \alpha > \frac{1}{2}$, then regular vertical lift of γ also belongs to $\mathrm{Hol}^\alpha([0,T],\mathbb{R}^{2n+1})$. Indeed, using Theorem A.1.3,

$$\begin{aligned}
|\,\mathrm{Z}(\gamma \llcorner [0,t]) - \mathrm{Z}(\gamma \llcorner [0,s]) + t - s| &= |\,\mathrm{Z}(\gamma \llcorner [s,t]) + t - s| \\
&\le |t-s| + |\,\mathrm{Z}(\gamma \llcorner [s,t]) - 2\mathcal{B}(\gamma(s), \gamma(t))| \\
&+ 2|\mathcal{B}(\gamma(s), \gamma(t))| \\
&\le |t-s| + C_\alpha \|\gamma\|^2_{\mathrm{hol}^\alpha} |t-s|^{2\alpha} + 2\|\gamma(s)\|\|\gamma(t) - \gamma(s)\| \\
&\le |t-s| + C_\alpha \|\gamma\|^2_{\mathrm{hol}^\alpha} |t-s|^{2\alpha} + C\|\gamma\|_\infty \|\gamma\|_{\mathrm{hol}^\alpha} |t-s|^\alpha \\
&\le C(\gamma)|t-s|^\alpha, \quad C(\gamma) < \infty.
\end{aligned}$$

Lemma 5.5.11. *Seen locally, level sets Γ of map $F \in C^{1,\alpha}_h(\mathbb{H}^n,\mathbb{R}^{2n})$, $\alpha > 0$, with surjective D_hF, are exactly regular vertical lifts of curves $\gamma \in \mathrm{Hol}^{\frac{1+\alpha}{2}}([0,T],\mathbb{R}^{2n})$.*

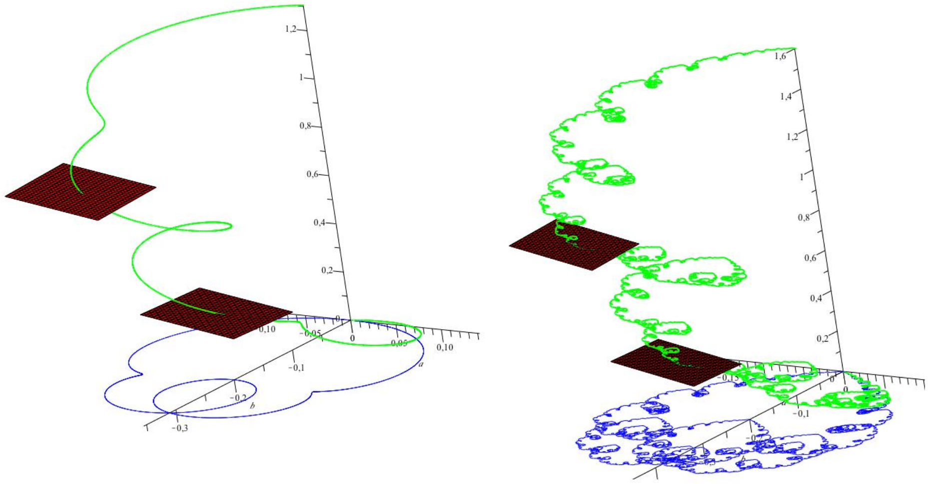

Figure 5.4.: Regular vertical lift for $n = 3$ (left) and $n = 16$ (right) of truncated Weierstrass curve $\{t \to \left(\sum_{i=1}^{n} 2^{-\frac{i}{2}} \sin(2\pi 2^i t), \sum_{i=1}^{n} 2^{-\frac{i}{2}}(1 - \cos(2\pi 2^i t))\right)\}$. The intersections of lifted curve with two horizontal planes are shown.

Proof. By Remark 5.3.9 the modulus of verticality of a vertical curve $\Gamma \subset F^{-1}(0)$ satisfies $\alpha(t) \lesssim t^{\alpha}$. Note also that by Corollary 5.5.6, Γ is ω-regular with $\omega(t) \lesssim t^{\alpha}$. This means, in particular, that Γ admits the natural parametrization by the "length", i. e. $\mathcal{H}^2 \llcorner \Gamma$. We see that $\pi(\Gamma) \in \mathrm{Hol}^{\frac{1+\alpha}{2}}$ w. r. t. this parametrization. Therefore, according to Eq. (5.23) curve Γ is a regular vertical lift of $\pi(\Gamma)$.

Take now $\gamma \in \mathrm{Hol}^{\frac{1+\alpha}{2}}([0,T], \mathbb{R}^{2n})$ and define Γ as a regular vertical lift of γ. By Theorem A.1.3, for any $t, s \in [0,T]$ close enough we have that

$$\begin{aligned} \mathsf{d}(\Gamma(t), \Gamma(s))^2 &\geq |t - s + \mathrm{Z}(\gamma \llcorner [s,t]) - 2\mathcal{B}(\gamma(t), \gamma(s))| \\ &\geq |t - s| - C_\alpha \|\gamma\|^2_{\mathrm{Hol}^{\frac{1+\alpha}{2}}} |t - s|^{1+\alpha} \\ &\geq C|t - s| \geq C \|\gamma\|^{-1}_{\mathrm{Hol}^{\frac{1+\alpha}{2}}} \|\gamma(t) - \gamma(s)\|^{\frac{2}{1+\alpha}}, \quad C = C(\gamma) > 0. \end{aligned}$$

To conclude we only have to apply Whitney extension Theorem 2.3.7 for $C_h^{1,\alpha}$-functions to $D_h F \llcorner \Gamma = \mathrm{Id}_{\mathbb{R}^{2n}}$ and $F \llcorner \Gamma = 0$. □

5.5.2. Euclidean Dimension.

Notation. The symbols $\mathcal{H}_E^{\alpha}$ and $\dim_E$ will stay for the Hausdorff measure and dimension w. r. t. the Euclidean metric on $\mathbb{H}^n$.

Remark 5.5.12. According to the comparison theorem[BTW09] between Euclidean and SubRiemannian dimension, $1 \leq \dim_E \Gamma \leq \dim \Gamma = 2$. Since the projection π is 1-Lipschitz (in Euclidean sense too), $\mathcal{H}_E^{\alpha}(\Gamma) \geq \mathcal{H}_E^{\alpha}(\pi(\Gamma))$ and $\dim_E \Gamma \geq \dim_E \pi(\Gamma)$. If Γ is 2-Ahlfors regular, then up to change the parametrization $\pi(\Gamma) \in \mathrm{hol}^{1/2}$ and, therefore, $\pi(\Gamma)$ has area zero : $\mathcal{L}^2(\pi(\Gamma)) = 0$.

Lemma 5.5.13. *The Euclidean dimension of vertical curves always belongs the closed interval* $[1,2]$ *and can take any value from it.*

Proof. We are going to make our construction in $\mathbb{H}^1$, the generalization to the higher dimensional Heisenberg groups are trivial. For $1 \le \beta < 2$ we can always find a "quasi-helix" curve of exponent β^{-1}, i. e. $\gamma\colon [0,1] \to \mathbb{R}^2$ such that

$$\|\gamma(t) - \gamma(s)\| \approx |t-s|^{\frac{1}{\beta}}, \quad s,t \in [0,1].$$

Such a curve can be explicitly built as Von Koch's auto-similar curve. Then, of course, $\dim_E \gamma = \beta$. Take Γ a regular vertical lift of γ (see Definition 5.5.9) that is, by Proposition 5.5.8, a vertical curve. By Remark 5.5.10, $\dim_E \Gamma \le \beta$, on the other hand, $\dim_E \Gamma \ge \dim_E \pi(\Gamma) = \beta$. To obtain the dimension $\beta = 2$ see example below. $\square$

Example 5.5.14. There is a 2-Ahlfors regular vertical curve $\Gamma \subset \mathbb{H}^1$ such that

$$\dim \Gamma = \dim_E \Gamma = \dim_E \pi(\Gamma) = 2.$$

Proof. The main idea is to build first a curve $\gamma\colon [0,1] \to \mathbb{R}^2$ of Euclidean dimension 2 such that $\|\gamma(t)-\gamma(s)\|^2 \le |t-s| m_\gamma(|t-s|)$ and m_γ satisfies Dini's condition Eq. (5.19). Then, we can take a vertical curve Γ as regular vertical lift of γ that has the wished properties. The curve γ will be obtained iteratively as Von Koch's curve with the factor of similarity that increases slowly at each iteration.

Let us fix a decreasing sequence $\{h_n\} \in (0, \frac{1}{2})$, $h_n \searrow 0$, and two points A_0^0 and A_1^0 in $\mathbb{R}^2$ with $l_0 := \|A_1^0 - A_0^0\| = 1$. We are going to define by dyadic iterations a sequence of the points $\{A_i^n\}$, $n \ge 0$ and $i = 0, \dots, 2^n$. To iterate, we replace each segment of n^{th}-generation by two segments $(n+1)^{\text{th}}$-generation as on Figure 5.5. We alternate the left and right sides (w. r. t. to old segments) on which new segments are added (so that new formed triangles lie inside the old ones). The length of each segment $[A_i^n, A_{i+1}^n]$ of n^{th}-generation is equal to $l_n = 2^{-n(\frac{1}{2}+h_n)}$. Let γ be a unique continuous function such

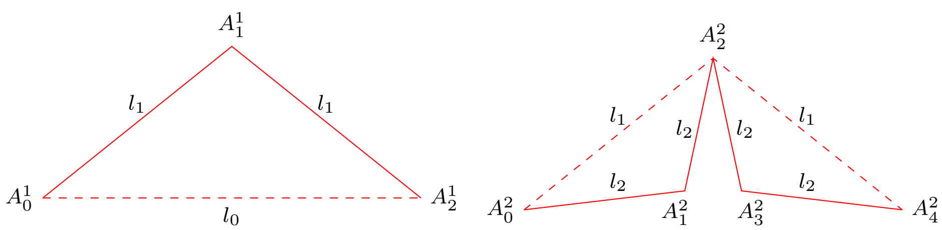

Figure 5.5.: First and second generation of dyadic points

that $\gamma(i/2^n) = A_i^n$ for every $n \ge 0$ and $i = 0, \dots, 2^n$. Observe that whatever $h_n \searrow 0$ we take, $\dim_E \gamma = 2$. In fact, it can be shown (compare to the proof of Theorem 5.3.5) that for every $\beta > 1/2$

$$\|\gamma(t) - \gamma(s)\| \ge |t-s|^\beta, \quad s,t \in [0,1].$$

For any couple of points A_i^n and A_j^n such that $0 < |j-i| < 2^{n-r}$, we can show by recurrence over $n = r+1, \dots,$ that

$$\|A_i^n - A_j^n\| \le 2 \sum_{k=r+1}^{n} l_k. \tag{5.25}$$

From Equation (5.25) we derive that m_γ satisfies $m_\gamma(2^{-r}) \le 2^r\big(2\sum_{k=r+1}^\infty l_k\big)^2$. In particular, $\gamma \in \mathrm{hol}^{1/2}$ if $kh_k \xrightarrow{k\to\infty} 0$. The series $\sum_{n=0}^\infty m_\gamma(2^{-n})$ is bounded (up to multiplicative and additive universal constants) by

$$\begin{aligned}\sum_{r=0}^\infty m_\gamma(2^{-r}) &\lesssim \sum_{r=0}^\infty 2^r\Big(\sum_{k=r+1}^\infty l_k\Big)^2 \\ &\le \sum_{r=0;\,j,k=1}^\infty 2^r l_k l_j \mathbf{1}_{(k\ge r+1, j\ge r+1)} \lesssim \sum_{j,k=1}^\infty 2^{\min\{k,j\}} l_k l_j \le \Big(\sum_{k=1}^\infty 2^{\frac{k}{2}} l_k\Big)^2.\end{aligned} \tag{5.26}$$

Therefore, if $\sum_n 2^{-nh_n} < \infty$, then $\sum_r m_\gamma(2^{-r}) < 0$ converges, i. e. m_γ satisfies Dini's condition.

Let us take, for instance, $h_n = (n\ln(n+1)^2+2)^{-1}$ and define $\gamma : [0,1] \to \mathbb{R}^2$ as above. Put $\kappa(t,s) = 2\big(\gamma_x(t)\gamma_y(s) - \gamma_y(t)\gamma_x(s)\big)$ then

$$|\kappa(s,q) + \kappa(q,t) - \kappa(s,t)| \lesssim m_\gamma(|t-s|)|t-s|, \quad q \in [s,t] \subset [0,1].$$

So, using Eq. (5.26) in Lemma 5.4.13, we see that the curve $t \to (\gamma(t), \mathrm{Z}(\gamma\llcorner[0,t]))$ exists and belongs to $\mathrm{hol}^{1/2}([0,1],\mathbb{H}^1)$ To conclude, we must now apply Proposition 5.5.8. $\square$

5.6. Irregular examples

As we have already mentioned flatness condition from Lemma 5.2.14 implies that $\dim\Gamma = 2$ but says nothing about the regularity of Hausdorff measure $\mathcal{H}^2$ on Γ. In this section we are going to develop the technique to be able to construct effectively vertical curves carrying irregular Hausdorff measure $\mathcal{H}^2$. Typically, we think about vertical curves having measure zero or infinity. Of course, those types of irregularities can be manifested only by very particular level sets and, certainly, does not reflect the generic behaviour. At least, we can assert this for vertical curves of infinite measure.

Remark 5.6.1. By the general coarea inequality for Lipschitz maps, see Theorem 2.1.13, the measure $\mathcal{H}^2(F^{-1}(a)\cap K) < \infty$ is finite for $\mathcal{L}^{2n}$-almost every $a \in \mathbb{R}^{2n}$ and every $K \Subset \mathbb{H}^n$.

Still, if we believe in coarea formula for $F \in \mathrm{Lip}(\mathbb{H}^n, \mathbb{R}^{2n})$, vertical curves of measure zero should be also rather exceptional.

Our construction technique is based on intrinsic characterization in Lemma 5.6.12 of the curve $\pi(\Gamma)$. To apply this characterization requires exact calculus of the "symplectic" Stieltjes sum (see Notation 5.5.2) for the curves γ with lower regularity (going beyond Lemma 5.4.13 and Theorem A.1.3). We are going to achieve this calculus for special family of γ given as *lacunary Fourier series*.

5.6.1. Stieltjes calculus for lacunary Fourier series

Notation 5.6.2. For $n \in \mathbb{Z}$ we introduce the basic functions

$$\begin{aligned}\phi_n(t) :=& (2\pi)^{-1}\exp(-2\pi I 2^n t),\\ \psi_n(t) :=& (2\pi)^{-1}\exp(2\pi I 2^n t).\end{aligned}$$

Let us also consider two series

$$
\begin{aligned}
f(t) &= \sum_{n=0}^{\infty} 2^{-\frac{n}{2}} \big(a_n\phi_n(t) + b_n\psi_n(t)\big), \quad a_n, b_n \in \mathbb{C},\\
g(t) &= \sum_{m=0}^{\infty} 2^{-\frac{m}{2}} \big(c_m\phi_m(t) + d_m\psi_m(t)\big), \quad c_m, d_m \in \mathbb{C}.
\end{aligned}
\tag{5.27}
$$

Observe that the functions f and g are real-valued if and only if $\bar{a}_n = b_n$ and $\bar{c}_m = d_m$. In all cases that we are interested in, we are going to assume that $f, g \in \bigcap_{\epsilon>0} \mathrm{Hol}^{1/2-\epsilon}([0,1],\mathbb{C})$.

For $h > 0$ we denote by $k(h)$ a unique integer such that $2^{-k(h)-1} \le h < 2^{-k(h)}$. For $h > 0$, we have the following classical upper bound

$$
\begin{aligned}
&\big|\sum_{n=0}^{\infty} 2^{-\frac{n}{2}} a_n(\phi_n(t+h) - \phi_n(t))\big|\\
&\le \sum_{n=0}^{k(h)} 2^{-\frac{n}{2}} |a_n| |\exp(2\pi I 2^n h) - 1| + \sum_{n=k(h)+1}^{\infty} 2^{-\frac{n}{2}} |a_n|\\
&\le \sum_{n=0}^{k(h)} 2^{-\frac{n}{2}} |a_n| h 2^n + 2^{-\frac{k(h)}{2}} \sum_{n=k(h)+1}^{\infty} 2^{-\frac{n-k(h)}{2}} |a_n|\\
&\le h^{\frac{1}{2}} L_{k(h)}(\{a_n\}),
\end{aligned}
$$

where we note

$$
L_k(\{a_n\}) := 2\big(\sum_{n=0}^{k} |a_n| 2^{-\frac{k-n}{2}} + \sum_{n=k+1}^{\infty} 2^{-\frac{n-k}{2}} |a_n|\big).
$$

Thus,

$$
|f(t+h) - f(t)|^2 \le 2h\big(L_{k(h)}(\{a_n\})^2 + L_{k(h)}(\{b_n\})^2\big).
$$

It is clear that $L_k(\{a_n\}) \lesssim \|a_n\|_\infty$, where $\|\cdot\|_\infty$ stands for the usual norm in a l^∞-space. Therefore, $\|f\|_{\mathrm{Hol}^{1/2}} \lesssim (\|a_n\|_\infty + \|b_n\|_\infty)$. Observe also that if $\lim_n a_n = \lim_n b_n = 0$ then $L_k(\{a_n\}) + L_k(\{b_n\}) \xrightarrow[k\to\infty]{} 0$ and, in this case, $f \in \mathrm{hol}^{1/2}$. Similar remarks are valid for the function g.

Fix $[s,t] \subset [0,1]$ and let $k^\star = k(t-s)$. Let $\sigma = \{s = t_0 < t_1 < \cdots < t_N < t_{N+1} = s\}$ be a subdivision of $[s,t]$. Our goal now is to estimate the Stieltjes sum $\int_\sigma f\,dg - f(s)(g(t) - g(s))$. We are going to pay attention to two different regimes:

1. $t - s = 1$ (with a convention that $k^\star = 0$);

2. $t - s \ll 1$.

We split this sum in two terms called diagonal (D_σ) and secondary (S_σ):

$$
D_\sigma = \sum_{n=0}^{\infty} 2^{-n} \big(a_n d_n \int_\sigma \tilde{\phi}_n\, d\psi_n + b_n c_n \int_\sigma \tilde{\psi}_n\, d\phi_n\big);
$$

$$S_\sigma = \sum_{n\neq m} 2^{-\frac{n+m}{2}} \big(a_n d_m \int_\sigma \tilde{\phi}_n \, d\psi_m + b_n c_m \int_\sigma \tilde{\psi}_n \, d\phi_m\big)$$
$$+ \sum_{n,m=0}^{\infty} a_n c_m 2^{-\frac{n+m}{2}} \int_\sigma \tilde{\phi}_n \, d\phi_m + b_n d_m 2^{-\frac{n+m}{2}} \int_\sigma \tilde{\psi}_n \, d\psi_m,$$

where $\tilde{\phi}_n(\cdot) = \phi_n(\cdot) - \phi_n(s)$, and $\tilde{\psi}_n(\cdot) = \psi_n(\cdot) - \psi_n(s)$.

Notation. We put $\Delta t_i := t_{i+1} - t_i$, $k_i = k(\Delta t_i)$, and $\|\sigma\| := \max_i |\Delta t_i|$.

Estimate of secondary term. Let us treat, for instance, the term $S_{a,c}$ relative to the coefficients $a_n c_m$. We also split this term in two ("small" and "big"):

$$S_{a,c} = S^s_{a,c} - S^b_{a,c} := \sum_{i=0}^{N} \sum_{n,m=0}^{\infty} a_n c_m 2^{-\frac{n+m}{2}} \int_{t_i}^{t_i+\Delta t_i} \big(\phi_n(t_i) - \phi_n(l)\big)\dot{\phi}_m \, dl$$
$$- \sum_{n,m=0}^{\infty} a_n c_m 2^{-\frac{n+m}{2}} \int_s^t \big(\phi_n(s) - \phi_n(l)\big)\dot{\phi}_m(l) \, dl.$$

Of course, formally speaking the last sum $S^b_{a,c}$ might not exist, but the reader should think about it as a finite sum up to K big enough and, then, when we get our estimates, make tend $K \to \infty$. We don't do this passage to the limit over K only in order to simplify the notations.

Depending on the size of Δt_i we can use one of the four following bounds that are quite easy to find:

$$\Big|\int_{t_i}^{t_i+\Delta t_i} \big(\phi_n(t_i) - \phi_n(l)\big)\dot{\phi}_m(l) \, dl\Big| \le \min \begin{cases} 2^{n+m}(\Delta t_i)^2; \\ 2^{n+1}\Delta t_i; \\ 2^m \Delta t_i; \\ 1. \end{cases} \tag{5.28}$$

For fixed $i \in [0, N]$, according to Eq. (5.28) we get

$$\sum_{n,m=0}^{\infty} |a_n c_m| \, 2^{-\frac{n+m}{2}} \Big| \int_{t_i}^{t_i+\Delta t_i} \big(\phi_n(t_i) - \phi_n(l)\big)\dot{\phi}_m(l) \, dl\Big|$$
$$\lesssim \sum_{n,m\le k_i} |a_n c_m| 2^{\frac{n+m}{2}} 2^{-2k_i} + \sum_{m>k_i\ge n} |a_n c_m| 2^{\frac{n-m}{2}} 2^{-k_i}$$
$$+ \sum_{n>k_i\ge m} |a_n c_m| 2^{\frac{m-n}{2}} 2^{-k_i} + \sum_{n,m>k_i} |a_n c_m| \, 2^{-\frac{n+m}{2}}$$
$$\le 2^{-k_i} \Big(\sum_{n=0}^{k_i} \frac{|a_n|}{2^{(k_i-n)/2}} \sum_{m=0}^{k_i} \frac{|c_m|}{2^{(k_i-m)/2}} + \sum_{n=0}^{k_i} \frac{|a_n|}{2^{(k_i-n)/2}} \sum_{m=k_i+1}^{\infty} \frac{|c_m|}{2^{(m-k_i)/2}}$$
$$+ \sum_{m=0}^{k_i} \frac{|c_m|}{2^{(k_i-m)/2}} \sum_{n=k_i+1}^{\infty} \frac{|a_n|}{2^{(n-k_i)/2}} + \sum_{n=k_i+1}^{\infty} \frac{|a_n|}{2^{(n-k_i)/2}} \sum_{m=k_i+1}^{\infty} \frac{|c_m|}{2^{(n-k_i)/2}} \Big).$$

We derive from it that

$$
\begin{aligned}
|S^s_{a,c}| &\lesssim \sum_{i=0}^{N} 2^{-k_i} L_{k_i}(\{a_n\}) L_{k_i}(\{c_m\}),\\
|S^b_{a,c}| &\lesssim 2^{-k^\star} L_{k^\star}(\{a_n\}) L_{k^\star}(\{c_m\}).
\end{aligned}
$$

Exactly in the same way we find an appropriate upper bound for the other terms of $S_\sigma = S^s_\sigma - S^b$, so, after that we end up with

$$
\begin{aligned}
|S^s_\sigma| &\lesssim \sum_{i=0}^{N} 2^{-k_i} \big(L_{k_i}(\{a_n\}) + L_{k_i}(\{b_n\})\big)\big(L_{k_i}(\{c_m\}) + L_{k_i}(\{d_m\})\big),\\
|S^b| &\lesssim 2^{-k^\star} \big(L_{k^\star}(\{a_n\}) + L_{k^\star}(\{b_n\})\big)\big(L_{k^\star}(\{c_m\}) + L_{k^\star}(\{d_m\})\big).
\end{aligned}
$$

Observe that $S^b = 0$ if $t - s = 1$ since each term in this sum will be zero in this case.

Estimate of diagonal term. We split the diagonal term as $D_\sigma = D^s_\sigma - D^b$ according to the difference $\int_\sigma f\,dg - f(s)(g(t) - g(s))$. Note also that $D^b_\sigma = 0$ for $t - s = 1$.

Recall that for our choice of ϕ_n and ψ_n,

$$
\begin{aligned}
(2\pi)^2 \phi_n(t_i)\big(\psi_n(t_{i+1}) - \psi_n(t_i)\big) &= \exp(2\pi I 2^n \Delta t_i) - 1,\\
(2\pi)^2 \psi_n(t_i)\big(\phi_n(t_{i+1}) - \phi_n(t_i)\big) &= \exp(-2\pi I 2^n \Delta t_i) - 1,
\end{aligned}
$$

and that for $x \in [0,1]$

$$
|\exp(xI) - 1 - Ix| \le \frac{\min\{x^2, 1\}}{2}.
$$

Therefore,

$$
\begin{aligned}
(2\pi)^2 D^s_\sigma &= \sum_{i=0}^{N} \sum_{n=0}^{\infty} 2^{-n} \big(a_n d_n (\exp(2\pi I 2^n \Delta t_i) - 1) + b_n c_n (\exp(-2\pi I 2^n \Delta t_i) - 1)\big)\\
&= 2\pi I \sum_{i=0}^{N} \sum_{n=0}^{k_i} \Delta t_i (a_n d_n - b_n c_n) + \tilde{S}^s_\sigma.
\end{aligned}
$$

and $\tilde{S}^s_\sigma$ can be bounded as follows

$$
\begin{aligned}
|\tilde{S}^s_\sigma| &\le \sum_{i=0}^{N} \sum_{n=k_i+1}^{\infty} 2^{-n+1}(|a_n d_n| + |b_n c_n|) + \sum_{i=0}^{N} \sum_{n=0}^{k_i} 2^{n-1} 2^{-2k_i}(|a_n d_n| + |b_n c_n|)\\
&\le \sum_{i=0}^{N} 2^{-k_i} \Big(\sum_{n=k_i+1}^{\infty} 2^{-n+k_i+1}(|a_n d_n| + |b_n c_n|) + \sum_{n=0}^{k_i} 2^{n-1-k_i}(|a_n d_n| + |b_n c_n|)\Big)\\
&\le \sum_{i=0}^{N} 2^{-k_i} \big(L_{k_i}(\{a_n\}) L_{k_i}(\{d_m\}) + L_{k_i}(\{c_m\}) L_{k_i}(\{b_n\})\big).
\end{aligned}
$$

In the same way, for the boundary term D^b we get

$$(2\pi)^2 D^b = 2\pi I(t-s)\sum_{n=0}^{k^\star}(a_n d_n - b_n c_n) + \tilde{S}^b,$$
$$|\tilde{S}^b| \le 2^{-k^\star}\big(L_{k^\star}(\{a_n\})L_{k^\star}(\{d_m\}) + L_{k^\star}(\{c_m\})L_{k^\star}(\{b_n\})\big).$$

Results. In resume, taking into account that $\sum_{i=0}^N \Delta t_i = t-s$, we obtain the following estimate

$$\int_\sigma f\,dg - f(s)(g(t)-g(s)) = (2\pi)^{-1} I \sum_{i=0}^N \Delta t_i \sum_{n=k^\star}^{k_i}(a_n d_n - b_n c_n) + R_\sigma^s + R^b, \tag{5.29}$$

where $R^b = 0$ if $t-s=1$, and,

$$|R_\sigma^s| \lesssim (t-s)\max_{k\ge k(\|\sigma\|)}\big(L_k(\{a_n\}) + L_k(\{b_n\})\big)\big(L_k(\{c_m\}) + L_k(\{d_m\})\big), \tag{5.30}$$
$$|R^b| \lesssim (t-s)\big(L_{k^\star}(\{a_n\}) + L_{k^\star}(\{b_n\})\big)\big(L_{k^\star}(\{c_m\}) + L_{k^\star}(\{d_m\})\big). \tag{5.31}$$

Now we reformulate this result in several different contexts.

Proposition 5.6.3. *Take $\|a_n\|_\infty + \|b_n\|_\infty < \infty$, $|c_m| + |d_m| \xrightarrow{m\to\infty} 0$ and the functions f and g given by Eq. (5.27). Then, for $t-s=1$*

$$\lim_{\|\sigma\|\to 0}\int_\sigma f\,dg = (2\pi)^{-1} I \sum_{n=0}^{\infty}(a_n d_n - b_n c_n), \tag{5.32}$$

if and only if the last series converges (not necessary absolutely).

Proof. Assume that the series converges. Since $\sum_{i=0}^N \Delta t_i = 1$,

$$\sum_{i=0}^N \Delta t_i \sum_{n=0}^{k_i}(a_n d_n - b_n c_n) - \sum_{n=0}^{\infty}(a_n d_n - b_n c_n) = \sum_{n=k(\|\sigma\|)}^{\infty}(a_n d_n - b_n c_n)\sum_{i=0}^N \Delta t_i(\mathbf{1}_{\{n\le k_i\}} - 1).$$

The last term tends to 0 when $\|\sigma\| \to 0$ because the function $n \to \sum_{i=0}^N \Delta t_i(\mathbf{1}_{\{n\le k_i\}} - 1)$ is monotone and bounded. □

Remark 5.6.4. The existence of Stieltjes integral $\int_0^1 f\,dg$ is equivalent to its existence on any other non-empty interval $[s,t] \subset [0,1]$. Indeed, due to similarity property of the lacunary series, the restriction of our integral on a smaller dyadic interval makes us forget about some finite number of regular terms that changes nothing for the convergence of series Eq. (5.32).

Proposition 5.6.5. *Take f and g as in Proposition 5.6.3. Then for a subdivision σ of any interval $[s,t] \subset [0,1]$*

$$\begin{aligned} &\Big| \int_\sigma f\,dg - f(s)(g(t)-g(s)) \\ &\qquad - (2\pi)^{-1} I \sum_\sigma \Delta t_i \sum_{n=k(|t-s|)}^{k(\Delta t_i)} (a_n d_n - b_n c_n)\Big| \le \omega(|t-s|)|t-s|, \end{aligned} \tag{5.33}$$

where $\omega(h) \to 0$ when $h \searrow 0$.

Proposition 5.6.6. *Take f and g real-valued functions given by Eq. (5.27). Assume that for all $n \ge 0$*

$$\max\{\|a_n\|_\infty, \|b_n\|_\infty, \|c_n\|_\infty, \|d_n\|_\infty\} \le K < \infty.$$

Then for a subdivision σ of any interval $[s,t] \subset [0,1]$

$$\begin{aligned} &\Big| \int_\sigma f\,dg - f(s)(g(t)-g(s)) \\ &\qquad - (2\pi)^{-1} I \sum_\sigma \Delta t_i \sum_{n=k(|t-s|)}^{k(\Delta t_i)} (a_n d_n - b_n c_n)\Big| \lesssim K^2|t-s|. \end{aligned} \tag{5.34}$$

Proposition 5.6.7. *Assume that $\|a_n\|_\infty + \|b_n\|_\infty < \infty$ and $|c_m| + |d_m| \xrightarrow{m\to\infty} 0$. Assume also that the corresponding functions $f \in \mathrm{Hol}^{\frac{1}{2}}$ and $g \in \mathrm{hol}^{1/2}$ are real-valued. Then for $t-s=1$ the limit values of $\int_\sigma f\,dg$ when $\|\sigma\| \to 0$ fills the closed interval*

$$\overline{\int_\sigma f\,dg}_{\|\sigma\|\to 0} = \big[\liminf_{n\to\infty} A_n ; \limsup_{n\to\infty} A_n\big], \quad A_n := \pi^{-1}\sum_{k=0}^{n} \mathrm{Im}(a_k d_k).$$

And if $\|a_n\|_\infty + \|b_n\|_\infty < \infty$ and $\|c_m\|_\infty + \|d_m\|_\infty < \infty$,

$$\overline{\int_\sigma f\,dg}_{\|\sigma\|\to 0} \subset \big[\liminf_{n\to\infty} A_n - l(f,g) ; \limsup_{n\to\infty} A_n + l(f,g)\big],$$

where $l(f,g) \lesssim (\|a_n\|_\infty + \|b_n\|_\infty)(\|c_m\|_\infty + \|d_m\|_\infty)$.

Proof. It suffices to distribute in the right way the convex weights Δt_i. □

Application: 1/2-Hölder curves in Heisenberg group.

Proposition 5.6.8. *A curve $\gamma \in \mathrm{Hol}^{1/2}([0,1], \mathbb{R}^2)$ can be obtained as the projection $\gamma = \pi(\lambda)$ of some curve $\lambda \in \mathrm{Hol}^{1/2}([0,1], \mathbb{H}^1)$ if and only if the partial sums there is a constant $K < \infty$ such that for any subdivision $\sigma = \{s = t_0 < t_1 < \cdots < t_{l+1} = t\}$ of any interval $[s,t] \subset [0,1]$*

$$\Big| \int_\sigma \gamma_x\,d\gamma_y - \gamma_x(s)(\gamma_y(t) - \gamma_y(s))\Big| \le K|t-s|. \tag{5.35}$$

Remark. As $\gamma \in \mathrm{Hol}^{1/2}([0,1],\mathbb{R}^2)$, the property Eq. (5.35) is equivalent by Remark A.1.4 to its "mixte" counterpart

$$\begin{aligned} &|\sum_\sigma \mathcal{B}(\gamma(t_i),\gamma(t_{i+1})) - \mathcal{B}(\gamma(s),\gamma(t))| \\ &= |\int_\sigma (\gamma_x\, d\gamma_y - \gamma_y\, d\gamma_x - \gamma_x(s)\gamma_y(t) + \gamma_x(t)\gamma_y(s))| \le K'|t-s|. \end{aligned} \tag{5.36}$$

Proof. First assume that $\lambda = (\gamma, z) \in \mathrm{Hol}^{1/2}([0,1],\mathbb{H}^1)$. If $K = \|\lambda\|_{\mathrm{Hol}^{1/2}}$, then summing up over σ the terms

$$-K^2(t_{i+1}-t_i) \le z(t_{i+1}) - z(t_i) - 2\mathcal{B}(\gamma(t_i),\gamma(t_{i+1})) \le K^2(t_{i+1}-t_i),$$

and subtracting

$$-K^2(t-s) \le z(t) - z(s) - 2\mathcal{B}(\gamma(s),\gamma(t)) \le K^2(t-s),$$

we get that

$$|\mathcal{B}(\gamma(s),\gamma(t)) - \sum_\sigma \mathcal{B}(\gamma(t_i),\gamma(t_{i+1}))| \le K^2(t-s).$$

Conversely, assume that $\gamma \in \mathrm{Hol}^{1/2}([0,1],\mathbb{R}^2)$ satisfies Eq. (5.35). Let us fix a sequence of subdivisions σ_n of the interval $[0,1]$ such that $\|\sigma_n\| \to 0$. Given σ_n, we define $\sigma_n([s,t])$ its restriction on $[s,t] \subset [0,1]$ as a subdivision containing s, t and all points of σ_n between s and t. The continuity of γ and the property Eq. (5.36) imply that the family of continuous functions $\{z_n\}_{n\ge 0}$ on $[0,1]$ defined as

$$z_n(t) := \sum_{\sigma_n([0,t])} \mathcal{B}(\gamma(t_i),\gamma(t_{i+1})),$$

is equi-continuous. The family $\{z_n\}_{n\ge 0}$ is clearly bounded, hence, it satisfies the hypothesis of the classical Arzelá–-Ascoli theorem, i. e. is precompact. Thus, we can take some limit point z of $\{z_n\}_{n\ge 0}$. By definition, for every $[s,t] \in [0,1]$ and some subsequence n_k

$$z(t) - z(s) = \lim_{k\to\infty} \sum_{\sigma_{n_k}([s,t])} \mathcal{B}(\gamma(t_i),\gamma(t_{i+1})).$$

Passing to the limit in Eq. (5.36), we get

$$|z(t) - z(s) - \mathcal{B}(\gamma(s),\gamma(t))| \le K'|t-s|,$$

so that, $(\gamma, 2z)$ will be an appropriate lift. □

Remark 5.6.9. If $\lambda \in \mathrm{Hol}^{1/2}([0,1],\mathbb{H}^1)$, then curve $t \to \exp(h(t)Z)(\lambda(t))$ is also in $\mathrm{Hol}^{1/2}([0,1],\mathbb{H}^1)$ if and only if $h\colon [0,1] \to \mathbb{R}$ is Lipschitz.

From Proposition 5.6.6 and Proposition 5.6.8 we deduce a rather simple criterion for lacunary series.

Corollary 5.6.10. *Take f and g real-valued functions given by Eq. (5.27). Assume that*

$$\max\{\|a_n\|_\infty, \|b_n\|_\infty, \|c_n\|_\infty, \|d_n\|_\infty\} \leq K_1 < \infty.$$

Then the curve $\gamma = \{t \to \big(f(t), g(t)\big)\}$ can be obtained as the projection $\gamma = \pi(\lambda)$ of some curve $\lambda \in \mathrm{Hol}^{1/2}([0,1], \mathbb{H}^1)$ if and only if the partial sums

$$|\sum_{k=0}^{N} \mathrm{Im}(a_k d_k)| \leq K_2 < \infty$$

are uniformly bounded.

Remark. Thus, not every $\gamma \in \mathrm{Hol}^{1/2}([0,1], \mathbb{R}^2)$ admits a lift to $\lambda \in \mathrm{Hol}^{1/2}([0,1], \mathbb{H}^1)$.

5.6.2. Rough vertical lift

We fix an increasing function $\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\epsilon(t) \searrow 0$ when $t \searrow 0$. We assume also that the function, defined as $h(t) = \epsilon(t)^{-1} t$ for $t > 0$ and $h(0) = 0$, is continuous on $\mathbb{R}_+$. Let $\gamma \colon [0,T] \to \mathbb{R}^{2n}$ be a continuous curve. Given ϵ and γ, we define a continuous function $\Delta z \colon [0,T]^2 \to \mathbb{R}$: for $0 \leq t, s \leq T$ we put

$$\Delta z(t,s) = h\big(\|\gamma(t) - \gamma(s)\|^2\big) + 2\mathcal{B}(\gamma(t), \gamma(s)). \tag{5.37}$$

For $t \in [0,T]$ we define a number, finite or infinite,

$$z(t) = \mathrm{Var}_{[0,t]}(\Delta z) = \sup\left\{\sum_{i=0}^{N} \Delta z(t_{i+1}, t_i) \;\middle|\; 0 = t_0 < t_1 < \cdots < t_{N+1} = t\right\}. \tag{5.38}$$

The choice of γ and ϵ determines therefore uniquely $z(t)$.

Definition 5.6.11. If $z(T) < \infty$, then the curve $\Gamma(t) = (\gamma(t), z(t)) \colon [0,T] \to \mathbb{H}^n$ is called the rough vertical lift of γ.

Lemma 5.6.12 (Characterization of $\pi(\Gamma)$)**.** *Let ϵ, γ, Δz and z be as above.*

1. *Assume that $z(T) < \infty$. Then $z \colon [0,T] \to \mathbb{R}$ is continuous and the rough vertical lift $\Gamma(t) = (\gamma(t), z(t))$ is a vertical curve.*

2. *The projection $\gamma = \pi(\Gamma)$ of any vertical curve $\Gamma \colon [0,T] \to \mathbb{H}^n$ admits a rough vertical lift for some ϵ.*

Remark 5.6.13 (relaxation of ϵ). For the existence of a rough vertical lift, the only important properties is that $\epsilon(t) \underset{t\to 0}{\longrightarrow} 0$ and $z_\epsilon(T) < \infty$. It is clear that $z_\epsilon(T) \leq z_{\tilde{\epsilon}}(T)$ if $\tilde{\epsilon} \geq \epsilon$. Now, if $z_\epsilon(T) < \infty$ for $\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$, $\epsilon(t) \underset{t\to 0}{\longrightarrow} 0$, then we can easily find continuous and monotone $\tilde{\epsilon} \geq \epsilon$, such that $\tilde{\epsilon}(t) \underset{t\to 0}{\longrightarrow} 0$, and $\tilde{\epsilon}(t)^{-1} t \underset{t\to 0}{\longrightarrow} 0$, that allows us to apply Lemma 5.6.12.

Proof. 1. Since the function Δz is continuous and the chain-variation $z(t)$ is bounded, then the chain-variation $z(t)$ is also continuous by Proposition A.3.5. It is clear that $\mathrm{Var}_{[0,t]}(\Delta z) \geq \mathrm{Var}_{[0,s]}(\Delta z) + \mathrm{Var}_{[s,t]}(\Delta z)$ for all $0 \leq s \leq t \leq T$. Hence,

$$z(t) - z(s) \geq \mathrm{Var}_{[s,t]}(\Delta z) \geq \Delta z(t,s),$$

that gives

$$\epsilon\big(\|\gamma(t) - \gamma(s)\|^2\big)\big(z(t) - z(s) - 2\mathcal{B}(\gamma(t), \gamma(s))\big) \geq \|\gamma(t) - \gamma(s)\|^2.$$

We put $a = \Gamma(t)$, $b = \Gamma(s)$. As ϵ is increasing,

$$\epsilon(c\mathsf{d}(a,b)^2)\mathsf{d}(a,b)^2 \geq \|\pi(a) - \pi(b)\|^2, \quad c = \epsilon(\mathrm{diam}(\gamma)^2) < \infty,$$

that is exactly the Whitney's condition (see Proposition 5.2.5) for the set $\Gamma([0,T])$.

2. If γ is a constant curve, we just take a part of the vertical axis above γ. Otherwise, for $0 < \delta \leq \mathrm{diam}(\gamma)^2$ we define

$$\epsilon(\delta) := \max\left\{ \frac{\|\gamma(t) - \gamma(s)\|^2}{\mathsf{d}(\Gamma(t), \Gamma(s))^2} \mid 0 \leq t, s \leq T,\ \|\gamma(t) - \gamma(s)\|^2 \leq \delta \right\}.$$

According to the Whitney's condition $\|\gamma(t)-\gamma(s)\|^2 = o(\mathsf{d}(\Gamma(t),\Gamma(s))^2)$ when $|t-s| \to 0$. That is why in the definition of ϵ the maximum is achieved and $\epsilon(\delta) > 0$ if $\delta > 0$. For $\delta_n \to 0$, take $t = t_n$ and $s = s_n$ that realise the value of $\epsilon(\delta_n)$ (i. e. argmax-points). By compactness, we can assume that $t_n \to t$, $s_n \to s$. If $t = s$, then $\lim \epsilon(\delta_n) = 0$ by the Whitney's condition. If $t \neq s$, then $\lim \epsilon(\delta_n) \leq \lim \delta_n \, \mathsf{d}(\Gamma(t), \Gamma(s))^{-2} = 0$ because we are working with an injective parametrisation of Γ. This shows that $\epsilon(\delta) \searrow 0$ when $\delta \searrow 0$.

By its definition, the function Δz built with those γ and ϵ satisfies $\Delta z(s,t) \leq z(\Gamma(t)) - z(\Gamma(s))$, $0 \leq s < t \leq T$. Therefore, $\mathrm{Var}_{[0,T]}(\Delta z) \leq z(\Gamma(T)) - z(\Gamma(0))$. Of course, we still need to slightly modify ϵ as in Remark 5.6.13. The rough vertical lift $\tilde{\Gamma}$ that we have obtained can be different from Γ that we started with. □

5.6.3. Examples construction

We are going to make our constructions in $\mathbb{H}^1$, the generalization on higher dimensions is trivial.

Proposition 5.6.14. *If $\{[0,1] \ni t \to \Gamma(t)\}$ is a vertical curve, then the Stieltjes sums for its projection $\gamma = \pi(\Gamma)$*

$$2\sum_{\sigma} \mathcal{B}\big(\gamma(t_{i+1}), \gamma(t_i)\big) \leq z(\Gamma(1)) - z(\Gamma(0)) < \infty$$

are uniformly bounded from above for all $\sigma = \{0 = t_0 < t_1 < \cdots < t_{N+1} = 1\}$. This condition is sufficient for a curve $\gamma \in \mathrm{hol}^{1/2}([0,1], \mathbb{R}^{2n})$ to be the projection $\gamma = \pi(\Gamma)$ of some vertical curve Γ.

Proof. Indeed, if γ is different from constant, we can put

$$\epsilon(\delta) = \max_{\|\gamma(t)-\gamma(s)\|^2 \geq \delta} \delta |t-s|^{-1}, \ \delta \in (0, \mathrm{diam}(\gamma)),$$

and apply Lemma 5.6.12. $\square$

Proposition 5.6.15. *If for a curve vertical* Γ*,*

$$\liminf_{\|\sigma\|\to 0} \sum_{\sigma} \|\pi(\Gamma_{i+1}) - \pi(\Gamma_i)\|^2 > 0,$$

then $\mathcal{H}^2(\Gamma) = \infty$*.*

Proof. Immediately follows from the verticality condition and area formula. $\square$

Example 5.6.16. There exists a vertical curve Γ such that $\mathcal{H}^2(\Gamma) = \infty$.

Proof. Let f and g be to real-valued functions given by Eq. (5.27). We choose the coefficients $\{a_n\}$ and $\{d_m\}$ in a such way that $\lim a_n = \lim d_m = 0$ and the sequence $\sum_{n=0}^{N} \mathrm{Im}(a_n d_n)$ tends to ∞ when $N \to \infty$ (one should pay attention to the sign). The curve $\gamma := (f, g) \in \mathrm{hol}^{1/2}([0,1], \mathbb{R}^2)$ and the Stieltjes sum

$$\sum_{\sigma} \mathcal{B}(\gamma(t_{i+1}), \gamma(t_i)) = \int_{\sigma} g\, df - \int_{\sigma} f\, dg$$

are uniformly bounded from above by Eq. (5.29) and Remark A.1.4. By Proposition 5.6.14, the curve γ admits a rough vertical lift Γ. Using area formula Corollary 5.4.16, Proposition 5.6.7 and Remark A.1.4 we obtain that

$$\mathcal{H}^2(\Gamma) = z(\Gamma(1)) - z(\Gamma(0)) + 2 \liminf_{\|\sigma\|\to 0} \Big(\int_{\sigma} f\, dg - \int_{\sigma} g\, df \Big) = \infty. \qquad \square$$

Example 5.6.17. There exists a vertical curve Γ such that $\mathcal{H}^2(\Gamma) = 0$.

Proof. Take two real-valued functions f and g in the form of Eq. (5.27) with $\lim b_n = \lim c_m = 0$ and define $\gamma := (f, g) \in \mathrm{hol}^{1/2}([0,1], \mathbb{R}^2)$. Our strategy is to find the coefficients of f and g in such a way that for any subdivision $\sigma = \{0 = t_0 < t_1 < \ldots < t_{N+1} = 1\}$,

$$\begin{aligned} \sum_{\sigma} \Delta z(t_{i+1}, t_i) &= \sum_{\sigma} h(\|\gamma(t_{i+1}) - \gamma(t_i)\|^2) + 2\mathcal{B}(\gamma(t_{i+1}), \gamma(t_i)) \\ &\leq 2 \limsup_{\|\tilde{\sigma}\|\to 0} \sum_{\tilde{\sigma}} \mathcal{B}(\gamma(t_{k+1}), \gamma(t_k)) < \infty, \end{aligned} \tag{5.39}$$

where $h(t) = t\epsilon(t)^{-1}$ as in Lemma 5.6.12 (a suitable ϵ will be specified later). Indeed, in this case γ admits a rough vertical lift Γ and according to Eq. (5.38) its z-component satisfies

$$z(1) - z(0) = 2 \limsup_{\|\tilde{\sigma}\|\to 0} \sum_{\tilde{\sigma}} \mathcal{B}(\gamma(t_{k+1}), \gamma(t_k)),$$

since the supremum can be reached by making $\|\sigma\| \to 0$. Moreover, by applying the area formula from Corollary 5.4.16 we will find that

$$\begin{aligned}\mathcal{H}^2(\Gamma) &= z(1) - z(0) + 2 \liminf_{\|\sigma\|\to 0} \sum_{\sigma} \mathcal{B}(\gamma(t_i), \gamma(t_{i+i})) \\ &= z(1) - z(0) - 2 \limsup_{\|\sigma\|\to 0} \sum_{\sigma} \mathcal{B}(\gamma(t_{i+1}), \gamma(t_i)) = 0.\end{aligned}$$

By Remark A.1.4 ($fg\big|_0^1 = 0$) and Proposition 5.6.7,

$$\limsup_{\|\tilde{\sigma}\|\to 0} \sum_{\tilde{\sigma}} \mathcal{B}(\gamma(t_{k+1}), \gamma(t_k)) = 2 \limsup_{\|\tilde{\sigma}\|\to 0} \int_{\tilde{\sigma}} g\, df = 2\pi^{-1} \limsup_{n\to\infty} \sum_{k=0}^{n} \operatorname{Im}(b_k c_k).$$

Using the notations of Section 5.6.1, we get

$$\begin{aligned}\sum_{\sigma} \mathcal{B}(\gamma(t_{i+1}), \gamma(t_i)) &= \int_{\sigma} g\, df - f\, dg \\ &= 2 \int_{\sigma} g\, df + r_\sigma = 2\pi^{-1} \sum_{\sigma} \Delta t_i \sum_{k=0}^{k_i} \operatorname{Im}(b_k c_k) + r_\sigma + R^s_\sigma,\end{aligned}$$

where

$$\begin{aligned}|r_\sigma| &\le \sum_{\sigma} \|\gamma(t_{i+1}) - \gamma(t_i)\|^2, \\ |R^s_\sigma| &\le C \sum_{\sigma} \Delta t_i \big(L_{k_i}(\{b_n\})^2 + L_{k_i}(\{c_m\})^2\big).\end{aligned}$$

Therefore, to fulfil Eq. (5.39), it will be sufficient that for any interval $[t_i, t_{i+1}]$,

$$\begin{aligned}&\frac{\|\gamma(t_{i+1}) - \gamma(t_i)\|^2}{\epsilon\big(\|\gamma(t_{i+1}) - \gamma(t_i)\|^2\big)} + 2\|\gamma(t_{i+1}) - \gamma(t_i)\|^2 + 2C\Delta t_i \big(L_{k_i}(\{b_n\})^2 + L_{k_i}(\{c_m\})^2\big) \\ &\qquad\qquad \le 4\pi^{-1} \Delta t_i \limsup_{n\to\infty} \sum_{k=k_i+1}^{n} \operatorname{Im}(b_k c_k) < \infty.\end{aligned} \tag{5.40}$$

Recall that we also have

$$\|\gamma(t_{i+1}) - \gamma(t_i)\|^2 \le 4\Delta t_i \big(L_{k_i}(\{b_n\})^2 + L_{k_i}(\{c_m\})^2\big).$$

It is convenient to take the sequences $\{b_n\}$ and $\{c_m\}$ that tend to 0 very slowly in such a way that

1. $\operatorname{Im}(b_l c_l) \ge 0$ for all $l \ge 0$;
2. for all $k \ge 0$,

$$4(C+4)\big(L_k(\{b_n\})^2 + L_k(\{c_m\})^2\big) \le \sum_{l=k}^{\infty} \operatorname{Im}(b_l c_l) < \infty;$$

3. $\lim\limits_{k\to\infty} \dfrac{L_k(\{b_n\})^2 + L_k(\{c_m\})^2}{\sum_{l=k}^{\infty} \operatorname{Im}(b_l c_l)} = 0.$

One can always find such sequences, for instance, if $-Ib_k = c_k = k^{-1}$ for k big enough, then $L_k(\{b_n\})^2 = L_k(\{c_m\})^2 \lesssim k^{-2}$ and $\sum_{l=k}^{\infty} \operatorname{Im}(b_l c_l) \gtrsim k^{-1}$. Given the sequences $\{b_n\}$ and $\{c_m\}$ satisfying the conditions above, we can produce a function $\epsilon(t) \underset{t\to 0}{\longrightarrow} 0$ that meets Eq. (5.40). Indeed, for $0 < \delta \le \operatorname{diam}(\gamma)^2$ we can define

$$\epsilon(\delta) = 2\delta \max_{\|\gamma(t)-\gamma(s)\|^2 \ge \delta} \left(|t-s| S_{k(|t-s|)}\right)^{-1}, \quad S_k := \sum_{l=k+1}^{\infty} \operatorname{Im}(b_l c_l).$$

Note that $\Delta t S_{k(\Delta t)}$ is strictly decreasing when $\Delta t \searrow 0$. That is why the value of $\epsilon(\delta)$ is achieved on the shortest interval $[\bar{s}, \bar{t}]$ such that $\|\gamma(\bar{t}) - \gamma(\bar{s})\|^2 = \delta > 0$, i. e.

$$\epsilon(\delta) = \frac{2\|\gamma(\bar{t}) - \gamma(\bar{s})\|^2}{|\bar{s} - \bar{t}| S_{k(|\bar{s}-\bar{t}|)}} \lesssim \frac{(L_{k(|\bar{s}-\bar{t}|)}(\{b_n\})^2 + L_{k(|\bar{s}-\bar{t}|)}(\{c_m\})^2)}{S_{k(|\bar{s}-\bar{t}|)}} \underset{\delta\to 0}{\longrightarrow} 0.$$

Thus, the inequality Eq. (5.40) is fulfilled:

$$\begin{aligned} &\frac{\|\gamma(t_{i+1}) - \gamma(t_i)\|^2}{\epsilon\big(\|\gamma(t_{i+1}) - \gamma(t_i)\|^2\big)} + 2\|\gamma(t_{i+1}) - \gamma(t_i)\|^2 + 2C\Delta t_i \big(L_{k_i}(\{b_n\})^2 + L_{k_i}(\{c_m\})^2\big) \\ &\qquad \le \frac{\Delta t_i S_{k_i}}{2} + 2(4+C)\Delta t_i \big(L_{k_i}(\{b_n\})^2 + L_{k_i}(\{c_m\})\big)^2 \\ &\qquad \le \frac{\Delta t_i S_{k_i}}{2} + \frac{\Delta t_i S_{k_i}}{2} = \Delta t_i S_{k_i}. \end{aligned}$$

A replacement of ϵ (by some $\tilde{\epsilon} \ge \epsilon$, according to Remark 5.6.13) can be needed, but this will make the last inequality only stronger. $\square$

Example 5.6.18. There exists a vertical curve vertical Γ such that $\operatorname{Var}^2(\pi(\Gamma)) = \infty$ w. r. t. the Euclidean distance on $\mathbb{R}^2$.

Proof. This time we are going to consider two real-valued functions f and g given by Eq. (5.27) such that the sequences $\{b_n\}$ and $\{c_m\}$ tend (in norm) slowly to infinity. By taking dyadic subdivisions, it is easy to show that in this case $\operatorname{Var}^2(\gamma) = \infty$ for $\gamma = (f, g)$. Remember that we still need to satisfy

$$\sum_{\sigma} \frac{\|\gamma(t_{i+1}) - \gamma(t_i)\|^2}{\epsilon\big(\|\gamma(t_{i+1}) - \gamma(t_i)\|^2\big)} + 2\mathcal{B}(\gamma(t_{i+1}), \gamma(t_i)) \le K < \infty,$$

for any subdivision σ of $[0,1]$ with some $\epsilon(t) \underset{t\to 0}{\longrightarrow} 0$ and some fixed constant K. Note that the first term of the last sum is positive and non-bounded that force the second term to go to $-\infty$.

Using a routine from Example 5.6.17, we see that it will be sufficient that for any interval $[t_i, t_{i+1}]$

$$\frac{\|\gamma(t_{i+1})-\gamma(t_i)\|^2}{\epsilon\big(\|\gamma(t_{i+1})-\gamma(t_i)\|^2\big)}+2\|\gamma(t_{i+1})-\gamma(t_i)\|^2+2C\Delta t_i\big(L_{k_i}(\{b_n\})^2+L_{k_i}(\{c_m\})^2\big)$$
$$\leq \Delta t_i\Big(-4\pi^{-1}\sum_{k=0}^{k_i+1}\mathrm{Im}(b_kc_k)+K\Big). \quad (5.41)$$

For the sequences $\{b_n\}$ and $\{c_m\}$ this condition can be interpreted as follows

1. $0<\dfrac{L_k(\{b_n\})^2+L_k(\{c_m\})^2}{-\sum_{l=0}^{k+1}\mathrm{Im}(b_lc_l)}\xrightarrow{k\to\infty}0;$
2. Addition technical condition: $-2^{-k}\sum_{l=0}^{k+1}\mathrm{Im}(b_lc_l)\searrow 0$ when $k\nearrow\infty$.

For instance, the sequences $Ib_l=c_l=l^\alpha$ will do for $\alpha>0$. The end of the proof is imitating the one of Example 5.6.17 (with $S_k=-\sum_{l=0}^{k+1}\mathrm{Im}(b_lc_l)$). $\square$

Example 5.6.19. There exists a 2-Ahlfors regular vertical curve Γ such that the metric density of $\mathcal{H}^2\llcorner\Gamma$ does not exist at every point: for every interior point $a\in\Gamma$

$$0<\liminf_{r\to 0}\frac{\mathcal{H}^2(\Gamma\cap B(a,r))}{2r^2}<\limsup_{r\to 0}\frac{\mathcal{H}^2(\Gamma\cap B(a,r))}{2r^2}=1.$$

Remark. We can replace here $2r^2$ by $\mathrm{diam}(\Gamma\cap B(a,r))^2$, see Eq. (5.14).

Proof. Take $\gamma=(f,g)\in\mathrm{hol}^{1/2}([0,1],\mathbb{R}^2)$ given by Eq. (5.27) such that the series

$$\sum_{n=0}^{N}a_nd_n-b_nc_n \quad (5.42)$$

diverges when $N\to\infty$ but stays bounded. By Corollary 5.6.10, there exists a lift $\lambda\in\mathrm{Hol}^{1/2}([0,1],\mathbb{H}^1)$, $\pi(\lambda)=\gamma$. By Proposition 5.5.8, the curve $\{t\to\Gamma(t)=\exp(CtZ)(\lambda(t))\}$ is vertical for $C>0$ big enough. Furthermore, this parametrisation is 1/2-bi-Hölder, i. e. $\mathsf{d}(\Gamma(s),\Gamma(t))^2\approx|t-s|$, and, therefore, Γ is 2-Ahlfors regular.

Take $0\leq s<t\leq 1$ with $|t-s|\to 0$. Let σ stands for a subdivision of $[s,t]$. By area formula Eq. (5.22), Proposition 5.6.5 and Remark A.1.4,

$$\begin{aligned}
&\frac{1}{2}\big(\mathsf{d}(\Gamma(s),\Gamma(t))^2-\mathcal{H}^2(\Gamma([s,t]))\big)\\
&\qquad=x_sy_t-y_sx_t-\liminf_{\|\sigma\|\to 0}\int_\sigma(x\,dy-y\,dx)\\
&\qquad=-\liminf_{\|\sigma\|\to 0}\pi^{-1}I\sum_\sigma\Delta t_i\sum_{n=k(|t-s|)}^{k(\Delta t_i)}(a_nd_n-b_nc_n)+o(|t-s|).
\end{aligned}$$

Note that

$$-\liminf_{\|\sigma\|\to 0}\sum_\sigma\Delta t_i\sum_{n=k(|t-s|)}^{k(\Delta t_i)}(a_nd_n-b_nc_n)=(t-s)\limsup_{N\to\infty}\sum_{n=k(|t-s|)}^{N}(b_nc_n-a_nd_n),$$

and, therefore, by 1/2-bi-Hölder equivalence,

$$\xi(s,t) := 1 - \frac{\mathcal{H}^2\big(\Gamma([s,t])\big)}{\mathsf{d}(\Gamma(s),\Gamma(t))^2} \approx \limsup_{N\to\infty} \sum_{n=k(|t-s|)}^{N} (b_n c_n - a_n d_n) + o(1).$$

So, because the series Eq. (5.42) does not converge, there is some sequence $|t_k - s_k| \to 0$ such that $\lim_{k\to\infty} \xi(s_k, t_k) > 0$. Observe also that there is always a sequence $|t_k - s_k| \to 0$ such that $\lim_{k\to\infty} \xi(s_k, t_k) = 0$.

To finish, note that the flatness and the 2-Ahlfors regularity of Γ implies that for any interior point $a \in \Gamma$

$$|\mathcal{H}^2(\Gamma \cap B(a,r)) - \mathcal{H}^2([c,b]_\Gamma)| = o(r^2), \quad r \to 0,$$
$$|\mathcal{H}^2(\Gamma \cap B(a,r)) - \mathcal{H}^2([c',b']_\Gamma)| = o(r^2), \quad r \to 0,$$

where $[c,b]_\Gamma$ the smallest interval containing $\Gamma \cap B(a,r)$ and $[c',b']_\Gamma$ is the largest interval contained in $\Gamma \cap B(a,r)$ (see also Proposition 5.4.17). □

Remark. The reader should not be surprised to see the exact value of $\limsup_{r\to 0}$ equal to 1 in Example 5.6.19. Indeed, if $\mathcal{H}^2(\Gamma) < \infty$ then by a general measure-theory argument in [Fed69, sec. 2.10.17–2.10.19], we have that

$$\limsup_{\delta\to 0} \left\{ \frac{\mathcal{H}^2(\Gamma \cap E)}{\operatorname{diam}(E)^2} \mid c \in E, \quad \operatorname{diam}(E) \le 0 < \delta \right\} = 1,$$

for $\mathcal{H}^2$-almost all point points $c \in \Gamma$. Recall also that due to the flatness of Γ, we can use only subintervals of Γ instead of an arbitrary set E. Moreover, observe that due to auto-similar nature of the lacunary Fourier series, there is no more dependence on the reference point c when we are calculating $\limsup_{r\to 0}$.

Example 5.6.20. There is a vertical curve $\Gamma \in \mathrm{Hol}^{1/2}([0,T], \mathbb{H}^1)$, $T > 0$, such that the metric derivative

$$\lim_{h\to 0} \frac{\mathsf{d}^2(\Gamma(t+h), \Gamma(t))}{h}$$

does not exist at every $t \in (0,T)$.

Proof. Take a curve Γ from Example 5.6.19 and reparametrize it by “lenght” $\mathcal{H}^2 \llcorner \Gamma$. □

Remark 5.6.21 (Lipschitz maps between graded groups). We can view $\lambda \in \mathrm{Hol}^{1/2}(\mathbb{R}, \mathbb{H}^1)$ as an image of Lipschitz map from an Abelian homogeneous group $\mathbf{R}_2 = (\mathbb{R}, |\cdot|^{\frac{1}{2}}, \delta_t)$, $\delta_t(a) = t^2 a$, (i. e. the snowflake on $\mathbb{R}$ of the exponent 1/2). In particular, Example 5.6.20 says that the Rademacher's theorem[1] is not valid for Lipschitz maps between two graded groups $\mathbf{R}_2$ and $\mathbb{H}^1$ (certainly because we have dropped the brackets generating condition for $\mathbf{R}_2$).

Question 5.6.22. Does there exist a vertical curve $\Gamma \subset \mathbb{H}^1$ such that $\mathcal{L}^2(\pi(\Gamma)) > 0$? We believe that answer is “YES”, but it is not sure that this can be done using lacunary Fourier series.

[1] named after Hans Rademacher it states the following. If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ is Lipschitz, then f is differentiable almost everywhere in U. Its generalization is valid for the Lipschitz maps between Carnot groups, see [Pan89].

6. Foliation by vertical curves

6.1. Maximal codimension case

The goal of this section is to show that the level sets of a map in $C^1_h(\mathbb{H}^n, \mathbb{R}^{2n})$ with surjective horizontal differential form locally a continuous foliation of $\mathbb{H}^n$.

Theorem 6.1.1. *Let $F \in C^1_h(\mathbb{H}^n, \mathbb{R}^{2n})$, $F(0) = 0$ and $D_h F(0)$ is sujective. Then there exists a homeomorphism*

$$[0,1] \times [-\delta, \delta]^{2n} \ni (t,p) \longrightarrow \Gamma_p(t) \in U \subset \mathbb{H}^n, \quad \delta > 0,$$

into some neighbourhood U of $0 \in \mathbb{H}^n$ such that for every $p \in [-\delta, \delta]^{2n}$,

1. $\Gamma_p([0,1]) = U \cap F^{-1}(p)$;
2. *the parametrization $\{[0,1] \ni t \to \Gamma_p(t)\}$ induces on the vertical curve $U \cap F^{-1}(p)$ an asymptotically optimally doubling measure (as in Lemma 5.4.3).*

Before we start the proof, we present some auxiliary results.

Notation 6.1.2. We denote by $d_h F(a)\colon \exp(H\mathbb{H}^n) \to \mathbb{R}^{2n}$ the restriction of $D_h F(a)$ on horizontal plane. Thus, if $D_h F(a)$ is surjective, $d_h F(a)$ is an isomorphism of linear spaces.

We are going to use the topological argument of Corollary 3.1.6. Note that in our case the only possible candidate for T is $\exp(H\mathbb{H}^n)$.

Proposition 6.1.3 (On parallel transport)**.** *Let $F \in C^1_h(\mathbb{H}^n, \mathbb{R}^{2n})$ such that $D_h F(a)$ is surjective for every $a \in \mathbb{H}^n$. Assume that $p, p' \in \mathbb{R}^{2n}$ and the norm of $\Delta p := p' - p$ is small. By Corollary 3.1.6, for $a, b \in F^{-1}(p)$ we can find $a' \in \exp(H\mathbb{H}^n)(a) \cap F^{-1}(p')$ and $b' \in \exp(H\mathbb{H}^n)(b) \cap F^{-1}(p')$. Then*

$$a^{-1} \cdot a' = \pi(a^{-1} \cdot a') = [d_h F(a)]^{-1} \langle \Delta p \rangle + o(|\Delta p|),$$
$$b^{-1} \cdot b' = \pi(b^{-1} \cdot b') = [d_h F(b)]^{-1} \langle \Delta p \rangle + o(|\Delta p|),$$

and, if $\mathsf{d}(a,b) = r$ is small enough,

$$|\mathsf{d}(a,b)^2 - \mathsf{d}(a',b')^2| \le |z(a^{-1} \cdot b) - z(a'^{-1} \cdot b')| \le |\Delta p| o(|\Delta p| + r),$$

where the error terms small-o are uniform as soon as points a, b, a', b' belongs to some compact part of $\mathbb{H}^n$.

Proof. The first two equations are rather obvious, so let us show the third one. The facts that $\mathsf{d}(a,b)$ is small enough and that a,b belongs to some vertical curve ensure that $\mathsf{d}(a,b)^2 = |z(a^{-1}\cdot b)|$. The same holds for a', b' because Δp is small too. Because $a^{-1}\cdot a'$ and $b^{-1}\cdot b'$ are horizontal elements we have

$$\begin{aligned}
|z(a^{-1}\cdot b) - z(a'^{-1}\cdot b')| = \\
&= |z(b) - z(b') + z(a') - z(a) - 2\mathcal{B}(\pi(a),\pi(b)) + 2\mathcal{B}(\pi(a'),\pi(b'))| \\
&= 2|\mathcal{B}(\pi(a'),\pi(b')) - \mathcal{B}(\pi(a),\pi(b)) + \mathcal{B}(\pi(b'),\pi(b)) - \mathcal{B}(\pi(a'),\pi(a))| \\
&= 2|\mathcal{B}(\pi(a') - \pi(b), \pi(b') - \pi(a))| \\
&= 2|\mathcal{B}(\pi(a') - \pi(a) + \pi(b) - \pi(b'), \pi(b') - \pi(b) + \pi(b) - \pi(a))| \\
&\lesssim |([d_hF(a)]^{-1} - [d_hF(b)]^{-1})\langle\Delta p\rangle + o(|\Delta p|)|\,||\Delta p| + o(r)|,
\end{aligned}$$

from where the conclusion follows. □

Corollary 6.1.4. *If in Proposition 6.1.3 we take $|\Delta p| \approx r$ small, then pairs a,b and a',b' sit in the same order on the vertical curves on which they lie.*

Proof. If, for instance, $0 < z(a^{-1}\cdot b) = r^2$ but $z(a'^{-1}\cdot b') < 0$, then it would give a contradiction: $r^2 = o(r^2)$. □

Continuous selection theorem. Let us recall one important topological result.

Theorem 6.1.5 ([Mic89]). *Let X be a paracompact space and Y a complete metric space. Take a lower-semi-continuous (l.s.c.) map $\phi\colon (X,d_X) \to (2^Y, \mathrm{dist}_{d_Y})$ such that $\phi(x) \subset Y$ is closed for all $x \in X$. Suppose that $\dim_{top} X \le n+1$, $\phi(x)$ is n-connected for all $x \in X$ and $\{\phi(x) \mid x \in X\}$ is locally equi-n-connected. Then ϕ admits a continuous selection, that is a continuous map $f\colon X \to Y$ such that $f(x) \in \phi(x)$ for all $x \in X$.*

A metric space M is n-connected if, for every $k \le n$, every continuous map from the k-sphere S^k to M is null homotopic (i. e. homotopic to a constant map). A collection $\mathcal{E} \subset 2^M$ is locally equi-n-connected if, for every $y \in \cup_{B\in\mathcal{E}} B$, every neighbourhood V of y in M contains a neighbourhood W of y in M such that, for all $B \in \mathcal{E}$ and $k \le n$, every map from S^k to $W \cap B$ is null-homotopic over $V \cap B$ (i. e. a homotopy taking values in $V \cap B$). The following classical example from [Mic56] should help the reader understand the definitions.

Example. Consider $\phi\colon [0,1] \to 2^{\mathbb{R}^2}$ defined by

$$\phi(x) = \begin{cases} \{(t, \sin(t^{-1})) \mid t \in [x/2, x]\}, & x \in (0,1]; \\ \{0\} \times [-1,1], & x = 0. \end{cases}$$

Observe that ϕ is l.s.c., $\phi(x)$ is homeomorphic to an interval, but the family $\{\phi(x) \mid x \in [0,1]\}$ is not equi-n-connected. By the way, if ϕ admitted a continuous selection, it would imply that the image $\bigcup_{x\in[0,1]} \phi(x)$ is arcwise connected, which is not true.

Proof of Theorem 6.1.1. The proof will be split in several parts.

Part I: Globalization. This part is necessary in order to provide an appropriate boundary condition for the homeomorphism $\Gamma_p(t)$ that we are going to construct.

Fix $\xi \in C^\infty(\mathbb{H}^n, \mathbb{R})$ such that $0 \le \xi \le 1$ and

$$\xi = \begin{cases} 1, & x \in B(0,1); \\ 0, & x \notin B(0,2). \end{cases}$$

For $R > 0$ we put $\xi_R = \xi \circ \delta_{1/R}$. Observe that $\|D_h \xi_R\| \lesssim R^{-1}$. Now we define a map $F_R = \xi_R F + (1 - \xi_R) D_h F(0)$. Its differential satisfies

$$D_h F_R(a) - D_h F(0) = \xi_R(a)(D_h F(a) - D_h F(0)) + (F(a) - D_h F(0)\langle a \rangle) D_h \xi_R(a).$$

By the definition of the differentiability of F,

$$\|\xi_R(a)(D_h F(a) - D_h F(0))\| \le \|D_h F(a) - D_h F(0)\| \le \omega_F(B(0, 2R)),$$

and for $a, b \in B(0, 2R)$

$$\|(F(a) - D_h F(0)\langle a \rangle) D_h \xi_R(a)\| \lesssim R^{-1}\|F(a) - D_h F(0)\langle a \rangle\| =$$
$$R^{-1}\|F(a) - F(0) - D_h F(0)\langle a \rangle\| \le R^{-1}\mathsf{d}(0,a)\omega_F(B(0, CR)) \lesssim \omega_F(B(0, CR)),$$

therefore, for some universal constant $C > 0$,

$$\max_{a \in \mathbb{H}^n} \|D_h F_R(a) - D_h F(0)\| \le C\omega_F(B(0, CR)) \to 0, \quad \text{when } R \to 0. \tag{6.1}$$

We can find $R_0 > 0$ small enough such that $2C\omega_F(B(0, CR_0)) \le n_F(\{0\})$. In particular, $n_{F_R}(\mathbb{H}^n) \ge n_F(\{0\})/2 > 0$ for $R \le R_0$, so that, $D_h F_R(a)$ is surjective for every $a \in \mathbb{H}^n$. What about the level sets of F_R in this case? We know that for each point $a \in F_R^{-1}(p)$, $p \in \mathbb{R}^{2n}$, there is a neighborhood of a, containing a ball $B(a, r_a)$, in which the level set $F_R^{-1}(p)$ is a vertical curve. Moreover, following the arguments to prove this fact (see Lemma 5.2.10, Proposition 5.2.12, and Theorem 5.3.7), we see that the radius r_a of this ball can be bounded from below by some $r^\star > 0$ depending only on $N_{F_R}(\mathbb{H}^n)$, $n_{F_R}(\mathbb{H}^n)$ and $\omega_{F_R}(\mathbb{H}^n)$ (with a dependence emphasized in Remark 5.2.11). By the way, Eq. (6.1) implies that when $R \le R_0$ the last three quantities can be controlled by their counterpart for F restricted on $B(0, CR)$. So, the radius r^* can be bounded from below independently of $R \le R_0$. Let us chose $R < \min\{r^\star/4, R_0\}$ (so that $B(0, 2R) \subset B(a, r^\star)$ for every $a \in B(0, 2R)$) and denote by $\tilde{F} := F_R$.

Of course, $F^{-1}(p) \cap B(0,R) = \tilde{F}^{-1}(p) \cap B(0,R)$ since F and $\tilde{F}$ coincide on $B(0,R)$. Obviously, outside $B(0, 2R)$, the level set $\tilde{F}^{-1}(p)$ is merely a left-translated vertical axis. Moreover, due to our choice of R, for any $p \in \tilde{F}(B(0, 2R))$, the two intersection points $\tilde{F}^{-1}(p) \cap \partial B(0, 2R)$ are connected by the vertical curve $\tilde{F}^{-1}(p) \cap B(0, 2R)$, and, thus, the whole level set $\tilde{F}^{-1}(p)$ is a vertical curve for any $p \in \mathbb{R}^{2n}$.

Part II: Continuity. Denote by $z_0 = (3R)^2$ and let us consider for $p \in \mathbb{R}^{2n}$ the truncated vertical curve $\Gamma_p := \tilde{F}^{-1}(p) \cap \{-z_0 \le z \le z_0\}$.

Let us introduce the metric space Y of all continuous maps $\gamma : [-1, 1] \to \mathbb{H}^n$ that

- are *vertically monotone* : $z(\gamma(s)^{-1} \cdot \gamma(t)) \geq 0$ for any $t \geq s$;
- satisfy the *boundary condition* : $z(\gamma(1)) = z_0$ and $z(\gamma(-1)) = -z_0$.

We endow Y with the sup metric $d_Y(\gamma_1, \gamma_2) = \sup_{t \in [-1,1]} \mathsf{d}(\gamma_1(t), \gamma_2(t))$ that turns it into a complete metric space. Let $X = (\mathbb{R}^{2n}, \|\cdot\|)$ and let us define a multivalued map $\phi : X \to (2^Y, \mathrm{dist}_{d_Y})$,

$$\phi(p) = \{\gamma \in Y \mid \gamma([-1,1]) = \Gamma_p\}.$$

Each set $\phi(p) \subset Y$ is obviously closed and, by the construction of $\tilde{F}$, not empty. The linear order on Γ_p naturally induces a partial order on $\phi(p)$. Observe also that $\phi(p)$ is n-connected for all $n \geq 0$, because if we fix one injective parametrization on Γ_p as a reference, the set $\phi(p)$ of all monotone parametrizations of Γ_p will be clearly convex w. r. t. it.

Proposition. *The map ϕ is bi-Lipschitz.*

Proof of Proposition. By Remark 3.1.11, the Hausdorff distance dist_{d_Y} between $\phi(p')$ and $\phi(p)$ is greater than $|p' - p| \operatorname{Lip}(\tilde{F})^{-1}$. By symmetry, we only need to prove that for any given $\gamma \in \phi(p)$ there is a $\gamma' \in \phi(p')$ such that $d_Y(\gamma, \gamma') \lesssim \Delta p := |p' - p|$. Of course, it is enough to prove it for small Δp.

For $a \in \Gamma_p$ let us denote by $a^\perp \in \Gamma_{p'} \cap \exp(H\mathbb{H}^n)(a)$ a point closest to a. From Corollary 3.1.6, we can derive that $\mathsf{d}(a, a^\perp) \leq K\Delta p$ with a constant K depending only on $\tilde{F}$. Choose a sequence of points $\{-1 = t_0 < t_1 < \ldots < t_N = 1\}$ such that $\mathsf{d}(\gamma(t_i), \gamma(t_{i+1})) \approx \Delta p$. Define $a_0 = \Gamma_{p'} \cap \{z = -z_0\}$, $a_N = \Gamma_{p'} \cap \{z = z_0\}$ and $a_i = \gamma(t_i)^\perp$ for $i = 1, \ldots, N-1$. If Δp is small enough, then by Corollary 6.1.4 the sequence $\{a_i\}$ respects the order on $\Gamma_{p'}$. To conclude we take any $\gamma' \in \phi(p')$ such that $\gamma'(t_i) = a_i$. $\square$

Let us check that the family $\{\phi(p) \mid p \in X\}$ is locally equi-n-connected for any $n \geq 0$. Fix $\gamma \in \phi(p)$. For $\epsilon > 0$ and $p' \in X$ we define two maps from $[-1,1]$ to $\Gamma_{p'}$ by

$$\gamma_{min}(t) = \min\{a \in \Gamma_{p'} \mid \mathsf{d}(a, \gamma(t)) \leq \epsilon\},$$
$$\gamma_{max}(t) = \max\{a \in \Gamma_{p'} \mid \mathsf{d}(a, \gamma(t)) \leq \epsilon\}.$$

By definition, the convex interval $I := \{\tilde{\gamma} \in \phi(p') \mid \gamma_{min} \leq \tilde{\gamma} \leq \gamma_{max}\} \subset Y$ contains the intersection $\phi(p') \cap B_Y(\gamma, \epsilon)$ and $d_Y(\gamma_i, \gamma) \leq \epsilon$ where $i \in \{min, max\}$. Here, the use of the metric d_Y still makes sense even if γ_{min} and γ_{max} are not necessarily continuous and may not belong to Y. By Corollary 5.3.13, $\mathrm{dist}_{d_Y}(\{\gamma\}, I) \lesssim \epsilon$, and therefore, $\{\phi(p) \mid p \in X\}$ is locally equi-n-connected, $n \geq 0$.

Thus, we are in a situation where we can apply the continuous selection Theorem 6.1.5 to the map ϕ. It provides us with a continuous map $\mathbb{R}^{2n} \ni p \to \Gamma_p(\cdot)$ such that $\Gamma_p(\cdot)$ is a parametrization of vertical curve Γ_p. The issue now is that this individual parametrization is just monotone and not necessarily injective. Probably, there is a purely topological argument that allows to get a globally continuous injective reparametrization in t of $(p,t) \to \Gamma_p(t)$. But we will explore metric properties of vertical curves in order to build an injective parametrization with fine metric properties.

Part III: Equi-doubling measures. Here, we are going to use the equi-vertical flatness of Γ_p in order to construct on it an asymptotically optimally doubling measure μ_p depending continuously on p.

Observe that the family of vertical curves $\{\Gamma_p\}_{p\in\mathbb{R}^{2n}}$ has some common modulus of verticality denoted by α. Without loss of generality, we can assume that α is continuous. We denote by $r(p)$ the radius given in Eq. (5.13) for $a = \Gamma_p(0)$ and $b = \Gamma_p(1)$. In particular, $r(p)$ is continuous w. r. t. p because $\Gamma_p(0)$ and $\Gamma_p(1)$ are so. We introduce two functions

$$s_+(p) = \inf\{t \mid \mathsf{d}(\Gamma_p(1), \Gamma_p(t)) \le r(p)\},$$
$$s_-(p) = \sup\{t \mid \mathsf{d}(\Gamma_p(0), \Gamma_p(t)) \le r(p)\},$$

and two closed subsets of $[0, 1] \times \mathbb{R}^{2n}$,

$$U_+ = \{(t, p) \mid t \ge s_+(p)\} \quad \text{and} \quad U_- = \{(t, p) \mid t \le s_-(p)\}.$$

By the choice of $r(p)$, $s_+(p) > s_-(p)$ for every p. By elementary topology, there is a continuous section separating U_+ and U_-, i. e. a continuous map $p \to s_{1/2}(p) \in [0, 1]$ such that $s_- < s_{1/2} < s_+$. Indeed, locally it is enough to check the separation by a constant section that is obvious because the complement of $(U_+ \cup U_-)$ is open.

For the sake of notation, we put also $s_0 \equiv 0$ and $s_1 \equiv 1$. In the same way, starting from the subfamilies of vertical curves

$$\{\Gamma_p(t) \mid t \in [s_0(p), s_{1/2}(p)]\}_p \quad \text{and} \quad \{\Gamma_p(t) \mid t \in [s_{1/2}(p), s_1(p)]\}_p$$

we obtain respectively continuous sections $s_{1/4}$ and $s_{3/4}$. We repeat this dyadic procedure to construct an ordered family of continuous sections $\{s_q \mid q \in \mathrm{dyadic}([0, 1])\}$.

The construction of the vertically monotone map $q \to s_q(p)$ is a little bit different from the procedure in the proof of Lemma 5.4.3, because each time we don't take an exact middle point. However, as it is shown in Remark 5.4.4, the measure μ_p induced on Γ_p by $q \to s_q(p)$ still enjoys the same estimate Eq. (5.16) (with m depending only on α). In particular, by Corollary 5.4.6, the map $q \to s_q(p)$ is bi-Hölder continuous (with constants independent of p), and so, admits a bi-Hölder continuous extension to $[0, 1]$. Thus, $(p, t) \to s_t(p)$ is a homeomorphism with the required properties (up to restriction to an appropriate neighbourhood of 0). □

6.2. Applications

6.2.1. Local topology via vertical foliation.

Corollary 6.2.1. *Let $F \in C^1_h(\mathbb{H}^n, \mathbb{R}^k)$, $1 \le k \le 2n$, with $F(0) = 0$ and $D_hF(0)$ surjective. Then in some neighbourhood of $0 \in \mathbb{H}^n$ the level set $F^{-1}(0)$ is locally homeomorphic to* Ker $D_hF(0)$*.*

Remark 6.2.2. The case of $k \le n$ is easy because locally a level set admits a local parametrization as an intrinsic graph (see [FSS07]).

Proof. The level sets in the case of $k = 2n$ correspond to vertical curves. Therefore, we only need to prove the theorem for $k \leq 2n-1$. We are going to use the fact that $F^{-1}(0)$ is foliated by vertical curves. Since $D_h F(0)$ is surjective, we can find a ("complementary") horizontal homomorphism $P\colon \mathbb{H}^n \to \mathbb{R}^{2n-k}$ such that map $\tilde{F} = (F, P)\colon \mathbb{H}^n \to \mathbb{R}^{2n}$ has surjective horizontal differential at 0. Let $\Gamma_p(t)\colon [-\delta, \delta]^{2n} \times [0,1] \to \mathbb{H}^n$ be the local foliation of $\mathbb{H}^n$ given by Theorem 6.1.1 with fibres $\Gamma_p = \tilde{F}^{-1}(p)$. Thus, $\Gamma_p(t)$ provides also a homeomorphism of $([-\delta, \delta]^{2n} \cap (P(\mathbb{H}^n), 0)) \times [0,1]$ to $F^{-1}(0) \cap U$, where U is some neighbourhood of 0. Note that $[-\delta, \delta]^{2n} \cap (P(\mathbb{H}^n), 0)$ is homeomorphic to a cube in $\mathbb{R}^{2n-k}$, so that, $F^{-1}(0)$ is locally homeomorphic to $\mathbb{R}^{2n-k+1}$. □

6.2.2. Irregular examples.

Remark 6.2.3. We can obtain easily a lower bound on the Hausdorff dimension of level sets of $F \in C^1_h(\mathbb{H}^n, \mathbb{R}^m)$, $1 \leq m \leq 2n$, using a general coarea inequality for Lipschitz maps. Assume that $F(0) = 0$ and $D_h F(0)$ is surjective. Take P as in Corollary 6.2.1. Then for $S := F^{-1}(F(0)) \cap U$, by the coarea inequality in Theorem 2.1.13, we get

$$\int_{\mathbb{R}^{2n-m}} \mathcal{H}^{2-\epsilon}(S \cap P^{-1}(p))\, d\mathcal{H}^{2n-m}(p) \lesssim \mathrm{Lip}(P)^{2n+2}\, \mathcal{H}^{2n+2-m-\epsilon}(S), \quad \epsilon \geq 0.$$

We know that $S \cap P^{-1}(p) = \tilde{F}(0, p)$ is a vertical curve that has the Hausdorff dimension 2. Using a topological argument similar to Lemma 3.1.2 one can show that the image $P(S)$ has positive $\mathcal{H}^{2n-m}$ measure. Thus, for any $\epsilon > 0$ the left-hand side integral is equal to $+\infty$, and, hence, $\dim S \geq 2n + 2 - m$.

Corollary 6.2.4. *For any integer $k \in [n+1, 2n]$ there exist maps $g_i \in C^1_h(\mathbb{H}^n, \mathbb{R}^k)$, $i = 1, 2$, with $g_i(0) = 0$ and $D_h g_i(0)$ surjective, and a neighbourhood U of $0 \in \mathbb{H}^n$ such that*

$$\mathcal{H}^{2n+2-k}(g_1^{-1}(0) \cap U) = \infty;$$
$$\mathcal{H}^{2n+2-k}(g_2^{-1}(0) \cap U) = 0.$$

Proof. Let Γ be an irregular vertical curve in $\mathbb{H}^1$, either with $\mathcal{H}^2(\Gamma) = 0$ or $\mathcal{H}^2(\Gamma) = \infty$ (see Section 5.6). Let $F \in C^1_h(\mathbb{H}^1, \mathbb{R}^2)$ be a map with surjective $D_h F$ such that $\Gamma \subset F^{-1}(0)$,

We define $g \in C^1_h(\mathbb{H}^n, \mathbb{R}^{2n+2-k})$ as follows

$$g(x_1, \ldots, x_n, y_1, \ldots, y_n, z) = (F(x_1, y_1, z), x_2, \ldots, x_n, y_2, \ldots, y_{k-n}).$$

It is clear that $D_h g$ is surjective. Now let us look at $g^{-1}(0)$. As a set it is the direct product of Γ and $\mathbb{R}^{2n-k} \ni (y_{k-n+1}, \ldots, y_{2n})$. The key point is that the metric induced on $g^{-1}(0)$ is nothing but a product metric. Indeed, because all coordinates $x_2 = \ldots = x_n = 0$ for every $a, b \in g^{-1}(0)$, all non-commutative contributions of form $2x_l y_l$, $2 \leq l \leq n$, to the z-coordinate of $a^{-1} \cdot b$ vanish.

- For the case of $\mathcal{H}^2(\Gamma) = 0$ we can apply classical results about product metrics.

Theorem ([Fed69, Th. 2.10.45]). *Let* (Γ, d) *be a metric space. Endow* $\Gamma \times \mathbb{R}^m$ *with a product metric*

$$d((a,x),(b,y))^2 = \mathsf{d}(a,b)^2 + \|y-y'\|^2, \quad a,b \in \Gamma, \quad y,y' \in \mathbb{R}^m.$$

Then $\mathcal{H}^{\alpha+m}(U \times V) \approx \mathcal{H}^{\alpha}(U)\mathcal{L}^m(V)$ *provided that* $\mathcal{H}^{\alpha}(U) < \infty$, $U \subset \Gamma$.

- For the case of $\mathcal{H}^2(\Gamma) = \infty$, we can use the coarea inequality Theorem 2.1.13 (recall that the projection on each factor is Lipschitz).

□

6.2.3. Coarea formula with extra regularity.

Theorem 6.2.5. *Let* $F \in C_h^{1,\alpha}(\mathbb{H}^n, \mathbb{R}^{2n})$, $\alpha > 0$. *Then for any Borel set* $E \subset \mathbb{H}^n$, *the coarea formula holds*

$$\int_{\mathbb{R}^{2n}} \mathcal{H}^2\big(F^{-1}(p) \cap E\big)\, d\mathcal{L}^{2n}(p) = K \int_E |\det d_h F(a)|\, d\mathcal{H}^{2n+2}(a), \tag{6.2}$$

where $K = K(\mathsf{d}, n, k)$ *is a geometric constant.*

Proof. First of all we should observe that Eq. (6.2) holds for any map F that is C^1 regular in usual sense. This can be done by applying the standard Euclidean coarea formula and next calculating the density of $\mathcal{H}^2$ w. r. t. $\mathcal{H}^1_{eucl}$ on level sets. See [Mag04] for an implementation of this strategy in Heisenberg groups, or [KV13] for the general situation. Observe next that by the coarea inequality [Mag02], we can ignore the characteristic set $\{a \in \mathbb{H}^n \mid \det d_h F(a) = 0\}$ and we may only consider the situation where $D_h F$ is surjective on $\overline{E}$.

We use the approximation of F by smooth maps $F_m \in C^\infty(\mathbb{H}^n, \mathbb{R}^{2n})$ such that $F_m \to F$ and $D_h F_m \to D_h F$ locally uniformly. One can get it by taking a standard convolution $F_m = F * \xi_m$ where ξ_m is an approximation of the Dirac mass δ_0. Since $|\det d_h F_m| \to |\det d_h F|$ locally uniformly on $\mathbb{H}^n$, the right-hand side of Eq. (6.2) converges. Therefore, it suffices to show that $\mathcal{H}^2\big(F_m^{-1}(p) \cap E\big)$ converges to $\mathcal{H}^2\big(F^{-1}(p) \cap E\big)$ for any $p \in \mathbb{G}^2$.

Note also that Eq. (6.2) is a local relation, it is enough to show it only, let say, for small open sets E. Thus, we can assume that level set $F^{-1}(p) \cap E$ (as well as $F_m^{-1}(p) \cap E$ for every m) is made as a union of vertical curves.

Since we require an extra regularity of F, its level sets are ω-regular, see Definition 5.5.1. Note that the same is true for level sets of F_m, essentially because ω depends on the modulus of continuity of horizontal derivatives and taking a convolution changes it in a bounded way. Thus, we can step away a little bit from the boundary of E and consider a compact part of $F^{-1}(p) \cap E$ given as a finite number of vertical curves without changing a lot the total $\mathcal{H}^2$-measure. We should recall also that the level sets of F_m converges to the level sets of F locally in E in the Hausdorff distance (see Definition 3.1.7 and Proposition 3.1.8). And now the conclusion follows from Proposition 6.2.6. □

The argument below reminds the uniform concentration principle (see [Dav05, Sec. 35]) that provides a sufficient condition for the lower semi-continuity of the Hausdorff measure w. r. t. the local convergence of sets in the Hausdorff distance. But because we require the equi strong Ahlfors regularity of the family of approximative vertical curves Γ_m and we know how the optimal (for $\mathcal{H}^2$) coverings are made on them, we can obtain an upper bound too.

Proposition 6.2.6. *Let $\{\Gamma_m\}$ a sequence of vertical curves having a common modulus of verticality α that converges to Γ in the Hausdorff distance. Then Γ is vertical curve with the same modulus of verticality. If, furthermore, vertical curves Γ_m are all ω-regular (with some fixed modulus ω) then $\mathcal{H}^2(\Gamma_m) \to \mathcal{H}^2(\Gamma)$ and, as a consequence, Γ is also ω-regular.*

Proof. By Golab's theorem, Γ is connected. Take two points $a, b \in \Gamma$ such that $a = \lim_k a_k$ and $b = \lim_k b_k$ for $a_k, b_k \in \Gamma_{m_k}$ and m_k a subsequence of m. Then,

$$\|\pi(a^{-1} \cdot b)\| = \lim_k \|\pi(a_k^{-1} \cdot b_k)\| \leq \lim_k \alpha(z(a_k^{-1} \cdot b_k))z(a_k^{-1} \cdot b_k) = \alpha(z(a^{-1} \cdot b))z(a^{-1} \cdot b),$$

so that, Γ is a vertical curve a modulus of verticality α.

Put $r_m := \mathrm{dist}_{\mathsf{d}}(\Gamma, \Gamma_m)$. Let $t > 0$ and $\{\min \Gamma = a_0 < a_1 < \ldots < a_N = \max \Gamma\}$ be a subdivision on Γ such that $\max_i \mathsf{d}(a_i, a_{i+1}) \leq t$. For $a_i \in \Gamma$ we chose $a_i^m \in \Gamma_m$ a closest to a_i point on Γ_m. By the triangle inequality, $\max_i |\mathsf{d}(a_i^m, a_{i+1}^m) - \mathsf{d}(a_i, a_{i+1})| \leq 2r_m$ It is easy to see that when r_m is small compared to $\min_i \mathsf{d}(a_i, a_{i+1})$ the sequence $\{a_i^m\}$ must respect the order on Γ_m. Note also that b_0^m and b_N^m are close to the corresponding end-points of Γ_m, that is, $\max\{\mathsf{d}(a_0^m, \min \Gamma_m), \mathsf{d}(a_N^m, \max \Gamma_m)\} = \tilde{r}_m \to 0$. Using the ω-regularity of Γ_m we obtain

$$\begin{aligned}
|\mathcal{H}^2(\Gamma_m) - \sum_i \mathsf{d}(a_i^m, a_{i+1}^m)^2| &\leq \sum_i |\mathsf{d}(a_i^m, a_{i+1}^m)^2 - \mathcal{H}^2([a_i^m, a_{i+1}^m]_{\Gamma_m})| \\
&\quad + \mathcal{H}^2([\min \Gamma_m, a_0^m]_{\Gamma_m}) + \mathcal{H}^2([a_N^m, \max \Gamma_m]_{\Gamma_m}) \\
&\lesssim \omega((t + 2r_m)^2) \sum_i \mathsf{d}(a_i^m, a_{i+1}^m)^2 + \tilde{r}_m^2 \\
&\lesssim \omega((t + 2r_m)^2)\mathcal{H}^2(\Gamma_m) + \tilde{r}_m^2.
\end{aligned}$$

We should add to this the following rather rough estimate:

$$|\sum_i \mathsf{d}(a_i^m, a_{i+1}^m)^2 - \mathsf{d}(a_i, a_{i+1})^2| \lesssim N r_m t.$$

Passing to the limit in m we get that

$$(1 - K\omega(t^2)) \limsup_m \mathcal{H}^2(\Gamma_m) \leq \sum_i \mathsf{d}(a_i, a_{i+1})^2 \leq (1 + K\omega(t^2)) \liminf_m \mathcal{H}^2(\Gamma_m),$$

with some $1 \leq K < \infty$. Hence, taking the limit when $t \to 0$ and using the area formula for vertical curves we obtain the conclusion. $\square$

A. Appendix

A.1. About Stieltjes integral

Here we recall some results concerning the theory of Stieltjes integral.

Definition A.1.1. Let $x, y \in C^0([0,T],\mathbb{R})$ be two continuous functions. The Stieltjes integral $\int_0^T x\,dy$ is defined as the limit (when it exists and is finite) of sums

$$\int_\sigma x\,dy := \sum_{i=0}^{l} x(t_i)\big(y(t_{i+1}) - y(t_i)\big),$$

over all subdivisions $\sigma := \{0 = t_0 < t_1 < \ldots < t_l < t_{l+1} = T\}$, when $\|\sigma\| := \max_i |t_{i+1} - t_i| \to 0$.

Remark. Since x and y are continuous, if we replace in Definition A.1.1 the summand $x(t_i)\big(y(t_{i+1}) - y(t_i)\big)$ by $x(\xi)\big(y(t_{i+1}) - y(t_i)\big)$, where ξ is any point from $[t_i, t_{i+1}]$, we obtain the same value of $\int_0^T x\,dy$.

Theorem ([Che05]). *Let $x, y \in C^0([0,T],\mathbb{R})$ be such that $\int_0^T x\,dy$ exists. Then it can be represented as follows*

$$\int_0^T x\,dy = \lim_{\tau\to 0+} \tau^{-1} \int_0^T \big(y(t+\tau) - y(t)\big)x(t)\,dt.$$

Theorem ([Smi25]). *If the Stieltjes integral $\int_0^1 \gamma_x\,d\gamma_y$ exists for a curve $\gamma = (\gamma_x,\gamma_y) \in C^0([0,1],\mathbb{R}^2)$, then*

$$\limsup_{\delta\to 0} \sum_{i=0}^{l} \mathcal{L}^2\big(\operatorname{ch}\{\gamma([t_i,t_{i+1}])\}\big) = 0, \tag{A.1}$$

where the supremum is taken over all subdivisions $\{0 = t_0 < t_1 < \ldots < t_l < t_{l+1} = 1\}$ with $\max_i |t_{i+1} - t_i| \le \delta$, and $\operatorname{ch}\{E\}$ stands for the convex hull of a set $E \subset \mathbb{R}^2$. In particular, γ has to be of area 0 in the plane, $\mathcal{L}^2\big(\gamma([0,1])\big) = 0$.

Remark A.1.2. Condition Eq. (A.1) is not sufficient for the existence of $\int \gamma_x\,d\gamma_y$. Indeed, note that Eq. (A.1) always holds if $\gamma \in \mathrm{hol}^{1/2}([0,1],\mathbb{R}^2)$. However, one can find an example of a curve $\gamma \in \mathrm{hol}^{1/2}$ for which $\int \gamma_x\,d\gamma_y$ does not exist (see Section 5.6.1).

Theorem ([Smi28]). *Let $\gamma \in \mathrm{hol}^{1/2}([0,1],\mathbb{R}^2)$ be a simple closed curve Let $D \subset \mathbb{R}^2$ be a bounded open set such that $\partial E = \gamma$. Then the Stieltjes integral exists and is equal to ($\pm$ according to the orientation)*

$$\pm \int_0^1 \gamma_x \, d\gamma_y = \mathcal{L}^2(D).$$

Remark. In general, the integral $\int \gamma_x \, d\gamma_y$ does not exist even for a simple closed curve (for instance, see [Bes55]).

Theorem A.1.3 ([Kon37; You36]). *Let $x \in \mathrm{Hol}^\alpha([0,T],\mathbb{R})$ and $y \in \mathrm{Hol}^\beta([0,T],\mathbb{R})$ with $\alpha + \beta > 1$. Then the integral $\int_0^T x \, dy$ exists in Stieltjes' sense, and, furthermore, for any $t \in [0,T]$,*

$$\Big| \int_0^T x \, dy - x(t)(y(T) - y(0)) \Big| \leq C_{\alpha+\beta} \|x\|_{\mathrm{Hol}^\alpha} \|y\|_{\mathrm{Hol}^\beta} T^{\alpha+\beta}.$$

The last result can be extended to more general moduli of continuity for f and g, see [You38; Bur48] and Lemma 5.4.13.

Remark A.1.4. Looking at the relation

$$\Big| \int_\sigma x \, dy + \int_\sigma y \, dx - xy\Big|_0^1 \Big| = \Big| \sum_{i=0}^{l} (x(t_{i+1}) - x(t_i))(y(t_{i+1}) - y(t_i)) \Big|, \tag{A.2}$$

we see that if $x, y \in \mathrm{Hol}^{1/2}([0,1])$, then for any sequence of subdivisions σ_n, $\|\sigma_n\| \underset{n\to\infty}{\longrightarrow} 0$,

$$\Big| \lim_{n\to\infty} \int_{\sigma_n} x \, dy + \lim_{n\to\infty} \int_{\sigma_n} y \, dx - xy\Big|_0^1 \Big| \leq \|x\|_{\frac{1}{2}} \|y\|_{\frac{1}{2}},$$

provided that the two limits exist. If, moreover, $x \in \mathrm{hol}^{1/2}$ or $y \in \mathrm{hol}^{1/2}$, then integration by parts is valid, i. e. the relation

$$\lim_{n\to\infty} \int_{\sigma_n} x \, dy + \lim_{n\to\infty} \int_{\sigma_n} y \, dx = xy\Big|_0^1,$$

holds assuming only that one of two limits exists.

A.2. Example of Carnot groups with rigid fibres

Let us consider an example of ultra-rigid nilpotent stratified Lie algebra taken from [DOW11, Ex. 3.6]. Its basis $\{X_1, \dots, X_{17}\}$ has the following stratification (of step three)

$$\mathfrak{g}' = \mathrm{span}\{X_1, \dots, X_{10}\} \oplus \mathrm{span}\{X_{11}, \dots, X_{16}\} \oplus \mathrm{span}\{X_{17}\}$$

with the following non-trivial commutators

$$\begin{aligned}
&[X_1, X_2] = X_{11}, && [X_1, X_3] = X_{13}, && [X_1, X_4] = X_{14},\\
&[X_1, X_5] = X_{15}, && [X_1, X_6] = X_{16}, && [X_2, X_3] = X_{13},\\
&[X_2, X_5] = X_{12}, && [X_2, X_6] = X_{14}, && [X_3, X_5] = X_{12},\\
&[X_3, X_6] = X_{13}, && [X_3, X_7] = X_{14}, && [X_4, X_5] = X_{12},\\
&[X_4, X_6] = X_{13}, && [X_4, X_8] = X_{14}, && [X_5, X_6] = X_{13},\\
&[X_5, X_8] = X_{12}, && [X_5, X_9] = X_{14}, && [X_6, X_8] = X_{12},\\
&[X_6, X_9] = X_{13}, && [X_6, X_{10}] = X_{14}, && [X_7, X_8] = X_{14},\\
&[X_7, X_9] = X_{12}, && [X_7, X_{10}] = X_{13}, && [X_8, X_9] = X_{13},\\
&[X_8, X_{10}] = X_{14}, && [X_9, X_{10}] = -X_{12}, && [X_1, X_{11}] = X_{17}.
\end{aligned}$$

The validity of the Jacobi identity in $\mathfrak{g}'$ is obvious. We also consider the quotient $\mathfrak{g} = \mathfrak{g}'/\operatorname{span}(X_{17}) \approx \operatorname{span}\{X_1, \ldots, X_{16}\}$, a nilpotent stratified Lie algebra of step two. The authors assert in [DOW11] that $\mathfrak{g}$ (as well as $\mathfrak{g}'$) is ultra-rigid and it can be verified by hand or using MAPLE software. Being ultra-rigid for a Lie algebra here means that its group of homogeneous automorphisms consists only of homogeneous dilations. Let us note that for Lemma A.2.1, it would be sufficient to find a Carnot Lie-algebra $\mathfrak{g}$ having (up to dilations) a discrete group of homogeneous automorphisms.

Lemma A.2.1. *Let $\mathbb{G}^1 = \exp(\mathfrak{g}')$ and $\mathbb{G}^2 = \exp(\mathfrak{g})$ be the Carnot groups associated with the stratified Lie algebra $\mathfrak{g}'$ and $\mathfrak{g}$. Let $F \in C^1_h(\Omega, \mathbb{G}^2)$ where $\Omega \subset \mathbb{G}^1$ is an open connected set. Assume that $e_1 \in \Omega$, $F(e_1) = e_2$ and $D_hF(e_1)$ is surjective, then $F = D_hF(e_1)$ on Ω. In particular, any level set of F is a subset of the left-translated center* $\exp(\operatorname{span}(X_{17}))$.

Proof. The ultra-rigidity of $\mathfrak{g}$ implies that there is a continuous map $t\colon \Omega \to \mathbb{R}_+$ such that $d_hF(a) = \exp_2^{-1} \circ D_hF(a) \circ \exp_1 \colon \mathfrak{g}' \to \mathfrak{g}$ satisfies

$$d_hF(a)\langle X_i \rangle = \delta_{t(a)}(X_i) \quad \text{for any} \quad i = 1, \ldots, 16,$$

and $d_hF(a)\langle X_{17}\rangle = 0$. Thus, we need to prove that $t(a) \equiv \mathit{const}$. Note that it is enough to prove it for Ω an open connected neighbourhood of e_1 on which D_hF is surjective.

We are going to use an elegant argument from [Pan89, Th. 4, p. 56] that has been used only for two-steps Carnot groups yet but fits also well $\mathfrak{g}'$ because there are only few non-zero commutators of order three. Let us recall it in detail.

We call any set of the form $\{\exp(sX)(a) \mid s \in [s_1, s_2] \subset \mathbb{R}\}$ a *X-horizontal segment* where X is a left-invariant horizontal vector field. For two horizontal vectors $X, Y \in \mathfrak{g}'_1$ and $t \in \mathbb{R}$ we consider a *quadrilateral* that is a quadruple of points linked by horizontal segments of the following form

$$\begin{aligned}
&(a,\\
&\ a \cdot \exp(X),\\
&\ a \cdot \exp(X) \cdot \exp(Y),\\
&\ a \cdot \exp(X) \cdot \exp(Y) \cdot \exp(tX)).
\end{aligned}$$

Assume that

$$[X, Y] \neq 0, \quad \text{and} \quad [X, [X, Y]] = [Y, [X, Y]] = 0. \tag{A.3}$$

With this assumption let us see under which condition the quadruple is horizontally closable (hc-quadruple), i. e. its last and first vertices can be connected by a horizontal segment. Projecting everything on horizontal level, we note that the only candidate for this segment is $\exp(-Y - (1+t)X)$ and using Baker-Campbell-Hausdorff formula we can write

$$\begin{aligned}
&\exp(X) \cdot \exp(Y) \cdot \exp(tX) \cdot \exp(-Y - (1+t)X) = \\
&\exp(X + Y + \frac{1}{2}[X, Y]) \cdot \exp(tX) \cdot \exp(-Y - (1+t)X) = \\
&\exp((1+t)X + Y + \frac{1}{2}[X, Y] - \frac{1}{2}t[X, Y]) \cdot \exp(-Y - (1+t)X) = \\
&\exp((1-t)\frac{1}{2}[X, Y]).
\end{aligned}$$

This implies that a quadruple is a hc-quadruple if and only if $t = 1$.

Let us take a hc-quadruple in Ω with X, Y satisfying Eq. (A.3). Since $D_h F(a)\langle\exp(Z)\rangle = \exp(t(a)X)$ for any horizontal $Z \in \mathfrak{g}_1$, by the Fundamental theorem of calculus, we obtain that the image by F of any Z-horizontal segment of Ω is a Z-horizontal segment in $\mathbb{G}^2$ (maybe of a different length). In particular, the image by F of any hc-quadruple in Ω is ahc-quadruple in $\mathbb{G}^2$. Obviously, if the hc-quadruple in the domain satisfies Eq. (A.3), then its image will do so as well. Thus, this implies that the lengths of horizontal segments $F(\{a \cdot \exp(sX) \mid s \in [0, 1]\})$ and $F(\{a \cdot \exp(X) \cdot \exp(Y) \cdot \exp(sX) \mid s \in [0, 1]\})$ must be the same. Taking the limit when $X \to 0$ in this equality we get that $d_h F(a)\langle X\rangle = d_h F(\exp(Y)(a))\langle X\rangle$, so that $t(a) = t(\exp(Y)(a))$.

We see that we can achieve the equality $t(a) = t(\exp(Y)(a))$ as soon as for a given horizontal vector Y, we can find some horizontal vector field X that enjoys Eq. (A.3). Note that in $\mathfrak{g}'$ for any horizontal vector Y from the basis we arrange this : for instance, for problematic $Y = X_1$ we cannot take $X = X_2$ but we can take $X = X_3$. Therefore, we obtain that the dilation factor t is constant along all horizontal segments. Due to bracket generating condition, moving along them (without getting out of Ω) we can always cover some neighbourhood of any interior point of Ω, and this finishes the proof. $\square$

A.3. About generalized variation

Definition A.3.1. Let $f\colon [0, 1]^2 \to \mathbb{R}$. We define the chain-variation of f on the interval $[s, t]$ as

$$\operatorname{Var}_{[s,t]} f := \sup \left\{ \sum_{i=0}^{l} f(t_{i+1}, t_i) \;\middle|\; s = t_0 < t_1 < \ldots < t_{l+1} = t \right\}.$$

Let us put $F(t) = \operatorname{Var}_{[0,t]} f$.

Remark A.3.2. As a function of intervals, $\operatorname{Var} f$ is, in general, only sub-additive. It is additive if, for instance, f satisfies the triangle inequality :

$$f(t,s) \le f(t,h) + f(h,s), \quad h \in [s,t].$$

Definition A.3.3. For a curve $\lambda\colon [0,1] \to (E,d)$ in a quasi-metric space and $p > 0$ we define the p-variation as follows

$$\operatorname{Var}^p(\lambda) = (\operatorname{Var} f)^{\frac{1}{p}} \quad \text{with} \quad f(t,s) = d(\lambda(t),\lambda(s))^p.$$

Proposition A.3.4. *The following statements hold for $p > 0$.*

1. *Any curve $\lambda \in \mathrm{Hol}^{\frac{1}{p}}$ has finite p-variation;*
2. *Any continuous curve of finite p-variation admits a reparametrization w. r. t. which its belongs to $\mathrm{Hol}^{\frac{1}{p}}$.*

Proof. 1. Obviously follows from definitions.

2. This reparametrization can be given by $t \to t + \operatorname{Var}^p(\gamma \llcorner [0,t])^p$.

□

Proposition A.3.5. *Let $f\colon [0,1]^2 \to \mathbb{R}$ be continuous and zero on the diagonal ($f(t,t) = 0$ for every $t \in [0,1]$). If the chain-variation $F(1) < \infty$ is finite, then it is also continuous, i. e. $F \in C^0([0,1],\mathbb{R})$.*

Proof. For $0 \le s \le t \le 1$,

$$F(t) \ge F(s) + \operatorname{Var}_{[s,t]} f \ge F(s) + f(t,s),$$

because one can take s as point of subdivision in the definition of $F(t)$. Since $f(t,t) = 0$,

$$\limsup_{s \nearrow t} F(s) \le F(t), \qquad \text{and } \liminf_{t \searrow s} F(t) \ge F(s).$$

As F is a bounded function, for any $\epsilon > 0$ one can find $\sigma = \{0 = t_0 < \ldots < t_n < t_{n+1} = t\}$ such that $F(t) \le \sum_\sigma f(t_{i+1},t_i) + \epsilon$. If $s \nearrow t$, we can assume that $s \in (t_n,t)$. Then we put $\sigma' = \{0 = t_0 < \ldots < t_n < s\}$ and

$$\begin{aligned} F(s) \ge \sum_{\sigma'} f(t_{i+1},t_i) &= f(s,t_n) - f(t,t_n) + \sum_\sigma f(t_{i+1},t_i) \\ &\ge f(s,t_n) - f(t,t_n) + F(t) - \epsilon. \end{aligned}$$

By continuity of f, $f(s,t_n) - f(t,t_n) \xrightarrow{s \to t} 0$. Therefore,

$$\liminf_{s \nearrow t} F(s) \ge F(t) - \epsilon, \quad \text{and hence,} \quad \lim_{s \nearrow t} F(s) = F(t).$$

By the same argument,

$$\lim_{t \searrow s} \operatorname{Var}_{[t,1]} f = \operatorname{Var}_{[s,1]} f, \quad \text{which implies that} \quad \lim_{t \searrow s} \operatorname{Var}_{[s,t]} f = 0.$$

For $t \searrow s$, we put $\sigma'' = \{0 = t_0 < \ldots < t_l < s \leq t_{l+1} < \ldots < t_{n+1} = t\}$, and

$$
\begin{aligned}
F(t) - \epsilon \leq& \sum_{\sigma} f(t_{i+1}, t_i) \\
&= f(t_{l+1}, t_l) - f(s, t_l) - f(t_{l+1}, s) + \sum_{\sigma''} f(t_{i+1}, t_i) \\
&\leq f(t_{l+1}, t_l) - f(s, t_l) - f(t_{l+1}, s) + F(s) + \mathrm{Var}_{[s,t]} f.
\end{aligned}
$$

By uniform continuity of f, $f(t_{l+1}, t_l) - f(s, t_l) - f(t_{l+1}, s) \to 0$ when $t \searrow s$. This gives $\limsup_{t \searrow s} F(t) \leq F(s)$, and, finally, $\lim_{t \searrow s} F(t) = F(s)$. $\square$

A.4. Non-uniqueness of projection of vertical curve on vertical axis

We show here an example of non-uniqueness of γ_z in Lemma 5.2.10.

Example A.4.1. Consider $A = \{0\} \cup \{a_n\}_{n=1}^{\infty} \cup \{b_k\}_{k=1}^{\infty} \Subset \mathbb{H}^1$, where $a_n = \frac{1}{n}(1, 0, 1)$ and $b_k = \frac{1}{k}(1, -\frac{1}{k}, 1)$. We are going to check by elementary calculus that the compact set A satisfies the Whitney condition Eq. (5.3) of Proposition 5.2.5.

Proof. Since 0 is the only limit point of $\mathcal{A}$, it is enough to check that for $n, k \to \infty$

1. $\|\pi(a_k) - \pi(a_n)\| = o\big(|z(a_n^{-1} \cdot a_k)|^{\frac{1}{2}}\big)$,
2. $\|\pi(b_k) - \pi(b_n)\| = o\big(|z(b_n^{-1} \cdot b_k)|^{\frac{1}{2}}\big)$,
3. $\|\pi(b_k) - \pi(a_n)\| = o\big(|z(a_n^{-1} \cdot b_k)|^{\frac{1}{2}}\big)$.

The first two cases are easy, so we will deal only with the third one, which is equivalent to

$$
\Big(\frac{1}{k} - \frac{1}{n}\Big)^2 + \frac{1}{k^4} = o\Big(\Big|\frac{1}{n} - \frac{1}{k} + 2\frac{1}{k^2 n}\Big|\Big).
$$

For fixed N large enough, let us estimate for $k \geq N$

$$
\sup_{n \geq N} \frac{(\frac{1}{k} - \frac{1}{n})^2 + \frac{1}{k^4}}{|\frac{1}{n} - \frac{1}{k} + 2\frac{1}{k^2 n}|} = \sup_{n \geq N} (s(k,n) + t(k,n)),
$$

where we denote by

$$
\begin{aligned}
s(k,n) :=& n k^{-2} |k(k-n) + 2|^{-1}, \\
t(k,n) :=& (k-n)^2 n^{-1} |k(k-n) + 2|^{-1}.
\end{aligned}
$$

It is easy to see that $s(k,n) \leq s(k,k) = \frac{1}{2k} \leq \frac{1}{2N}$. To get an upper bound for $b(k,n)$ we put $g(x) = \frac{(x-k)^2}{x}$ and $h(x) = k(k-x) + 2$. One has

$$
\left(\frac{g(x)}{h(x)}\right)' = -\frac{(x-k)(x(k^2-2) - (k^3+2k))}{x^2(kx - 2 - k^2)}.
$$

The critical points of $g(x)/h(x)$ are the following (in increasing order):

$$x_1 = 0 < N, \quad x_2 = k \geq N, \quad x_3 = k + \frac{2}{k}, \quad x_4 = \frac{(k^2+2)}{k^2-2} < k+1.$$

In particular, $g(x)/h(x)$ is decreasing on $(0, k)$ and increasing on $(k+1, \infty)$. Therefore, $\sup_{n \geq N} t(k, n) \leq \max\{t(k, N), t(k, \infty)\} = \max\{t(k, N), \frac{1}{k}\}$. It is enough to estimate $\sup_{k \geq N} t(k, N)$. The same calculation shows that the function $\frac{(x-N)^2}{N(x(x-N)+2)}$ is increasing for $x \geq N$, so $\sup_{k \geq N} t(k, N) = t(\infty, N) = N^{-1}$. $\square$

A.5. Norm of conjugate element

Let $a, b \in \mathbb{G}$. We want to estimate here a homogeneous norm $\rho(k)$ of conjugate element $k = a^{-1} \cdot b \cdot a$. More precisely, we want to show that

$$\rho(k) \leq C(a) \rho(b)^{1/\deg(\mathbb{G})},$$

where $C(a)$ is bounded on compact subsets (that is in fact of polynomial growth). Let us see the objects on Lie algebra's side: we put $a = \exp(X)$, $b = \exp(Y)$, $k = \exp(Z)$. Then the adjoint action reads

$$Z = \sum_{n=0}^{\deg \mathbb{G}} \frac{1}{n!} \underbrace{[X, [X, \ldots, [X, Y], \ldots]]}_{n+1 \text{ brackets}}.$$

Therefore, using comparison theorem between linear and homogeneous norm (Proposition 2.1.7), it is sufficient to produce the following estimate

$$\rho(k)^{\deg(\mathbb{G})} \lesssim \|Z\| \leq \sum_{n=0}^{\deg(\mathbb{G})} \frac{1}{n!} \|X\|^n \|Y\| \leq \|Y\| exp(\|X\|) \lesssim \rho(b) exp(\rho(a)),$$

where *exp* stands for the numerical exponential function.

Bibliography

[BCC14] F. Bigolin, L. Caravenna, and F. Serra Cassano. "Intrinsic Lipschitz graphs in Heisenberg groups and continuous solutions of a balance equation". In: *Annales de l'Institut Henri Poincare (C) Non Linear Analysis* (2014).

[BK14] Francesco Bigolin and Artem Kozhevnikov. "Tangency, Paratangency and Four-Cones Coincidence Theorem in Carnot Groups". In: *Journal of Convex Analysis* 21.3 (2014).

[CM14] Vasilis Chousionis and Pertti Mattila. "Singular integrals on self-similar sets and removability for Lipschitz harmonic functions in Heisenberg groups". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2014.691 (2014), pp. 29–60.

[CMPS14] Giovanna Citti, Maria Manfredini, Andrea Pinamonti, and Francesco Serra Cassano. "Smooth approximation for intrinsic Lipschitz functions in the Heisenberg group." In: *Calc. Var. Partial Differ. Equ.* 49.3-4 (2014), pp. 1279–1308.

[BL13] Costante Bellettini and Enrico Le Donne. "Regularity of sets with constant horizontal normal in the Engel group." In: *Commun. Anal. Geom.* 21.3 (2013), pp. 469–507.

[DR13] Enrico Le Donne and Tapio Rajala. *Assouad dimension, Nagata dimension, and uniformly close metric tangents*. 2013.

[FMS13] Bruno Franchi, Marco Marchi, and Raul Paolo Serapioni. *Intrinsic Lipschitz graphs within Carnot groups*. 2013.

[HT13] Tuomas Hytönen and Olli Tapiola. *Almost Lipschitz-continuous wavelets in metric spaces via a new randomization of dyadic cubes*. 2013.

[KV13] M. Karmanova and S. Vodopyanov. "A coarea formula for smooth contact mappings of Carnot-Carathéodory spaces". In: *Acta Appl. Math.* 128 (2013), pp. 67–111.

[Mag13] Valentino Magnani. "Towards Differential Calculus in stratified groups". In: *J. Aust. Math. Soc.* 95.1 (2013), pp. 76–128.

[HM12] David Herron and Daniel Meyer. "Quasicircles and bounded turning circles modulo bi-Lipschitz maps". In: *Rev. Mat. Iberoam.* 28.3 (2012), pp. 603–630.

[MV12] Roberto Monti and Davide Vittone. "Sets with finite $\mathbb{H}$-perimeter and controlled normal." In: *Math. Z.* 270.1-2 (2012), pp. 351–367.

[DOW11] Enrico Le Donne, Alessandro Ottazzi, and Ben Warhurst. *Ultrarigid tangents of sub-Riemannian nilpotent groups.* Apr. 2011.

[Big10] F. Bigolin. "Intrinsic Regular Hypersurfaces and PDEs, the Case of the Heisenberg Group," PhD thesis. LAP Lambert Academic Publishing, 2010.

[BS10a] Francesco Bigolin and Francesco Serra Cassano. "Distributional solutions of Burgers' equation and intrinsic regular graphs in Heisenberg groups". In: *J. Math. Anal. Appl.* 366.2 (2010), pp. 561–568.

[BS10b] Francesco Bigolin and Francesco Serra Cassano. "Intrinsic regular graphs in Heisenberg groups vs. weak solutions of non-linear first-order PDEs". In: *Adv. Calc. Var.* 3.1 (2010), pp. 69–97.

[FSS10] B. Franchi, R. Serapioni, and F. Serra Cassano. "Differentiability of intrinsic Lipschitz functions within Heisenberg groups". In: *Journal of Geometric Analysis* (2010), pp. 1–41.

[Koz10] A. D. Kozhevnikov. "Inverse and implicit function theorems on Carnot manifolds". In: *Sibirsk. Mat. Zh.* 51.6 (2010), pp. 1322–1339.

[LM10] Gian Paolo Leonardi and Valentino Magnani. *Intersections of intrinsic submanifolds in the Heisenberg group.* Sept. 2010.

[AKL09] Luigi Ambrosio, Bruce Kleiner, and Enrico Le Donne. "Rectifiability of sets of finite perimeter in Carnot groups: existence of a tangent hyperplane." In: *J. Geom. Anal.* 19.3 (2009), pp. 509–540.

[BTW09] Zoltán M. Balogh, Jeremy T. Tyson, and Ben Warhurst. "Sub-Riemannian vs. Euclidean dimension comparison and fractal geometry on Carnot groups". In: *Adv. Math.* 220.2 (2009), pp. 560–619.

[CM09] Vasilis Chousionis and Pertti Mattila. *Singular integrals on Ahlfors-David regular subsets of the Heisenberg group.* Sept. 2009.

[MSC09] Pertti Mattila, Raul Serapioni, and Francesco Serra Cassano. "Characterizations of intrinsic rectifiability in Heisenberg groups". Feb. 2009.

[DH08] J. Dydak and J. Higes. "Asymptotic cones and Assouad-Nagata dimension". In: *Proc. Amer. Math. Soc.* 136.6 (2008), pp. 2225–2233.

[Vit08] Davide Vittone. "Submanifolds in Carnot groups". PhD thesis. Edizioni della Normale, Pisa, 2008.

[CDPT07] Luca Capogna, Donatella Danielli, Scott D Pauls, and Jeremy Tyson. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem.* Vol. 259. Springer Science & Business Media, 2007.

[DG07] S. Dolecki and G.H. Greco. "Towards historical roots of necessary conditions of optimality : Regula of Peano". In: *Control and Cybernetics* 36 (2007), pp. 491–518.

[FSS07] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. "Regular submanifolds, graphs and area formula in Heisenberg groups". In: *Adv. Math.* 211.1 (2007), pp. 152–203.

[LCL07] Terry J. Lyons, Michael Caruana, and Thierry Lévy. *Differential equations driven by rough paths*. Vol. 1908. Lecture Notes in Mathematics. Berlin: Springer, 2007.

[LV07] Terry Lyons and Nicolas Victoir. "An extension theorem to rough paths". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24.5 (2007), pp. 835–847.

[Vod07] S. K. Vodop'yanov. "Geometry of Carnot-Carathéodory spaces and differentiability of mappings". In: *The interaction of analysis and geometry*. Vol. 424. Contemp. Math. Providence, RI: Amer. Math. Soc., 2007, pp. 247–301.

[ASV06] Luigi Ambrosio, Francesco Serra Cassano, and Davide Vittone. "Intrinsic regular hypersurfaces in Heisenberg groups". In: *J. Geom. Anal.* 16.2 (2006), pp. 187–232.

[CM06a] G. Citti and M. Manfredini. "Implicit function theorem in Carnot-Carathéodory spaces". In: *Commun. Contemp. Math.* 8.5 (2006), pp. 657–680.

[CM06b] G. Citti and M. Manfredini. "Implicit function theorem in Carnot-Carathéodory spaces." In: *Commun. Contemp. Math.* 8.5 (2006), pp. 657–680.

[Daf06] Constantine M. Dafermos. "Continuous solutions for balance laws". In: *Ric. Mat.* 55.1 (2006), pp. 79–91.

[FL06] Denis Feyel and Arnaud de La Pradelle. "Curvilinear integrals along enriched paths". In: *Electron. J. Probab.* 11 (2006), no. 34, 860–892 (electronic).

[FSC06] B. Franchi, R. Serapioni, and F.S. Cassano. "Intrinsic Lipschitz graphs in Heisenberg groups". In: *Journal of Nonlinear and Convex Analysis* 7.3 (2006), pp. 423–441.

[Jea06] Frédéric Jean. "On Hausdorff Measures of Curves in Sub-Riemannian Geometry". In: *Preprint du CMAP, École Polytechnique* 11 (2006).

[VP06] S. K. Vodop'yanov and I. M. Pupyshev. "Whitney-type theorems on the extension of functions on Carnot groups". In: *Sibirsk. Mat. Zh.* 47.4 (2006), pp. 731–752.

[CKL05] Luca Capogna, Carlos E. Kenig, and Loredana Lanzani. *Harmonic measure*. Vol. 35. University Lecture Series. Providence, RI: American Mathematical Society, 2005.

[Dav05] Guy David. *Singular sets of minimizers for the Mumford-Shah functional.* Progress in Mathematics 233. Basel: Birkhäuser. xiv, 581 p., 2005.

[FSS05] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. "Intrinsic submanifolds, graphs and currents in Heisenberg groups". In: *Lecture notes of Seminario Interdisciplinare di Matematica. Vol. IV.* Lect. Notes Semin. Interdiscip. Mat., IV. S.I.M. Dep. Mat. Univ. Basilicata, Potenza, 2005, pp. 23–38.

[LS05] Urs Lang and Thilo Schlichenmaier. “Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions”. In: *Int. Math. Res. Not.* 58 (2005), pp. 3625–3655.

[MM05] Roberto Monti and Daniele Morbidelli. “Regular domains in homogeneous groups.” In: *Trans. Am. Math. Soc.* 357.8 (2005), pp. 2975–3011.

[War05] B. Warhurst. “Jet spaces as nonrigid Carnot groups”. In: *J. Lie Theory* 15.1 (2005), pp. 341–356.

[AS04] Andrei A. Agrachev and Yuri L. Sachkov. *Control theory from the geometric viewpoint*. Vol. 87. Encyclopaedia of Mathematical Sciences. Berlin: Springer-Verlag, 2004.

[Che05] V. Chernyatin. “A new necessary condition for existence of the Stieltjes integral”. In: *Real Anal. Exchange* 30.1 (2004–2005), pp. 207–215.

[Mag04] Valentino Magnani. “Note on coarea formulae in the Heisenberg group”. In: *Publ. Mat.* 48.2 (2004), pp. 409–422.

[Fal03] Kenneth Falconer. *Fractal geometry*. Second. Hoboken, NJ: John Wiley & Sons Inc., 2003.

[FSS03a] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. “On the structure of finite perimeter sets in step 2 Carnot groups”. In: *J. Geom. Anal.* 13.3 (2003), pp. 421–466.

[FSS03b] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. “On the structure of finite perimeter sets in step 2 Carnot groups.” In: *J. Geom. Anal.* 13.3 (2003), pp. 421–466.

[FSS03c] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. “Regular hypersurfaces, intrinsic perimeter and implicit function theorem in Carnot groups”. In: *Comm. Anal. Geom.* 11.5 (2003), pp. 909–944.

[LQ02] Terry Lyons and Zhongmin Qian. *System control and rough paths*. Oxford Mathematical Monographs. Oxford: Oxford University Press, 2002.

[Mag02] Valentino Magnani. “On a general coarea inequality and applications”. In: *Ann. Acad. Sci. Fenn. Math.* 27.1 (2002), pp. 121–140.

[Mon02] Richard Montgomery. *A tour of subriemannian geometries, their geodesics and applications*. Vol. 91. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, 2002.

[DKT01] Guy David, Carlos Kenig, and Tatiana Toro. “Asymptotically optimally doubling measures and Reifenberg flat sets with vanishing constant”. In: *Comm. Pure Appl. Math.* 54.4 (2001), pp. 385–449.

[Roh01] Steffen Rohde. “Quasicircles modulo bilipschitz maps”. In: *Rev. Mat. Iberoamericana* 17.3 (2001), pp. 643–659.

[DT99] G. David and T. Toro. “Reifenberg flat metric spaces, snowballs, and embeddings”. In: *Math. Ann.* 315.4 (1999), pp. 641–710.

[Kor98] Michael Brian Korey. “Ideal weights: asymptotically optimal versions of doubling, absolute continuity, and bounded mean oscillation”. In: *J. Fourier Anal. Appl.* 4.4-5 (1998), pp. 491–519.

[Luu98] Jouni Luukkainen. “Assouad dimension: Antifractal metrization, porous sets, and homogeneous measures.” In: *J. Korean Math. Soc.* 35.1 (1998), pp. 23–76.

[Lyo98] Terry J. Lyons. “Differential equations driven by rough signals”. In: *Rev. Mat. Iberoamericana* 14.2 (1998), pp. 215–310.

[Gro96] Mikhael Gromov. “Carnot-Carathéodory spaces seen from within”. In: *Sub-Riemannian geometry.* Vol. 144. Progr. Math. Basel: Birkhäuser, 1996, pp. 79–323.

[Mic89] E. Michael. “A generalization of a theorem on continuous selections”. In: *Proc. Amer. Math. Soc.* Vol. 105. 1989, pp. 236–243.

[Pan89] Pierre Pansu. “Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un”. In: *Ann. of Math. (2)* 129.1 (1989), pp. 1–60.

[FS82] G. B. Folland and Elias M. Stein. *Hardy spaces on homogeneous groups.* Vol. 28. Mathematical Notes. Princeton, N.J.: Princeton University Press, 1982.

[Gro81] Mikhael Gromov. “Groups of polynomial growth and expanding maps. Appendix by Jacques Tits.” In: *Publ. Math., Inst. Hautes Étud. Sci.* 53 (1981), pp. 53–78.

[TV80] P. Tukia and J. Väisälä. “Quasisymmetric embeddings of metric spaces.” In: *Ann. Acad. Sci. Fenn., Ser. A I, Math.* 5 (1980), pp. 97–114.

[Ass79] Patrice Assouad. “Étude d’une dimension métrique liee à la possibilité de plongements dans $\mathbb{R}^n$”. In: *C. R. Acad. Sci., Paris, Sér. A* 288 (1979), pp. 731–734.

[Llo78] N. G. Lloyd. *Degree theory.* Cambridge: Cambridge University Press, 1978.

[Var74] Veeravalli S Varadarajan. *Lie groups, Lie algebras, and their representations.* Vol. 102. Prentice-Hall Englewood Cliffs, NJ, 1974.

[Fed69] Herbert Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[AF65] JP Aubin and H. Frankowska. “Set-Valued Analysis. 1990”. In: *Birkhäuser, Boston* (1965).

[Rei60] E. R. Reifenberg. “Solution of the Plateau Problem for m-dimensional surfaces of varying topological type”. In: *Acta Math.* 104 (1960), pp. 1–92.

[Mic56] E. Michael. “Continuous selections. I”. In: *The Annals of Mathematics* 63.2 (1956), pp. 361–382.

[Bes55] A. S. Besicovitch. “On limits of the area of a polygon inscribed in a simple closed curve”. In: *Quart. J. Math. Oxford Ser. (2)* 6 (1955), pp. 135–142.

[Bur48] J. C. Burkill. “Differential properties of Young-Stieltjes integrals”. In: *J. London Math. Soc.* 23 (1948), pp. 22–28.

[Dyn47] E. B. Dynkin. “Calculation of the coefficients in the Campbell-Hausdorff formula”. In: *Doklady Akad. Nauk SSSR (N.S.)* 57 (1947), pp. 323–326.

[Cho40] Wei-Liang Chow. “Über Systeme von liearren partiellen Differentialgleichungen erster Ordnung”. In: *Mathematische Annalen* 117.1 (1940), pp. 98–105.

[Ras38] PK Rashevsky. “Any two points of a totally nonholonomic space may be connected by an admissible line”. In: *Uch. Zap. Ped. Inst. im. Liebknechta, Ser. Phys. Math* 2 (1938), pp. 83–94.

[You38] L. C. Young. “General inequalities for Stieltjes integrals and the convergence of Fourier series”. In: *Math. Ann.* 115.1 (1938), pp. 581–612.

[Kon37] V. Kondurar. “Sur l’intégrale de Stieltjes”. In: *Rec. Math. Moscou* 2(44).2 (1937), pp. 361–366.

[You36] L. C. Young. “An inequality of the Hölder type, connected with Stieltjes integration”. In: *Acta Math.* 67.1 (1936), pp. 251–282.

[Smi28] H. L. Smith. “On relative content and Green’s lemma”. In: *Trans. Amer. Math. Soc.* 30.2 (1928), pp. 405–419.

[Smi25] H. L. Smith. “On the existence of the Stieltjes integral”. In: *Trans. Amer. Math. Soc.* 27.4 (1925), pp. 491–515.